Numerical Methods

Unit - III

by

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Lagrange's Interpolation (For Unequal intervals):

Let $y_0, y_1, \ldots y_n$ be n+1 points of a function y = f(x) where f(x) is assumed to be a polynomial in x, corresponding to arguments $x_0, x_1, \ldots x_n$, not necessarily equally spaced. Then

$$y = f(x) = \frac{(x-x_1)(x-x_2)....(x-x_n)}{(x_0-x_1)(x_0-x_2)....(x_0-x_n)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)} y_1$$

$$.........$$

$$+ \frac{(x-x_0)(x-x_1)....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)....(x_n-x_{n-1})} y_n$$

This is called the *Lagrange's formula for interpolation.*

Problem: 1

Using Lagrange's interpolation formula, find the value of y corresponding to $x = 10$ from the following data.

$x:$	5	6	9	11
$y:$	12	13	14	16

Solution:

Given $x_0 = 5, \quad x_1 = 6, \quad x_2 = 9, \quad x_3 = 11$

$y_0 = 12, \quad y_1 = 13, \quad y_2 = 14, \quad y_3 = 16$

By Lagrange's interpolation,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Using x = 10, we get,

$$y(10) = \frac{4 \times 1 \times (-1)}{(-1)(-4)(-6)}(12) + \frac{5 \times 1 \times (-1)}{1(-3)(-5)}(13) + \frac{5 \times 4 \times (-1)}{4 \times 3 \times (-2)}(14) + \frac{5 \times 4 \times 1}{6 \times 5 \times 2}(16)$$

$\therefore y(10) = 14.67$

Problem: 2

Use Lagrange's interpolation formula to find the value of y at $x = 6$, given the data.

x :	3	7	9	10
y :	168	120	72	63

Solution:

Given the data

$$x_0 = 3, \quad x_1 = 7, \quad x_2 = 9, \quad x_3 = 10$$

$$y_0 = 168, \quad y_1 = 120, \quad y_2 = 72, \quad y_3 = 63$$

By Lagrange's interpolation,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Using x = 6, we get,

$$y(6) = \frac{(-1)(-3)(-4)}{(-4)(-6)(-7)}(168) + \frac{3\times(-3)\times(-4)}{4(-2)(-3)}(120)$$
$$+ \frac{3\times(-1)\times(-4)}{6\times 2\times(-1)}(72) + \frac{3\times(-3)\times(-1)}{7\times 3\times 1}(63)$$

$\therefore y(6) = 147$

Problem: 3

Apply Lagrange's formula to find $f(5)$, given that $f(1)=2$, $f(2)=4$, $f(3)=8$ and $f(7)=128$.

Solution:

Given the data

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 3, \quad x_3 = 7$$
$$y_0 = 2, \quad y_1 = 4, \quad y_2 = 8, \quad y_3 = 128$$

By Lagrange's interpolation,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Using x = 5, we get,

$$f(5) = \frac{3\times 2\times(-2)}{(-1)(-2)(-6)}(2) + \frac{4\times 2\times(-2)}{1\times(-1)(-5)}(4)$$
$$+ \frac{4\times 3\times(-2)}{2\times 1\times(-4)}(8) + \frac{4\times 3\times 2}{6\times 5\times 4}(128)$$

$\Rightarrow f(5) = 38.8$

Problem: 4

Given $u_0 = 6$, $u_1 = 9$, $u_3 = 33$ and $u_7 = -15$. Find u_2

Solution:

Given the data

$$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7$$

$$y_0 = 6, y_1 = 9, y_2 = 33, y_3 = -15$$

By Lagrange's interpolation,

$$\mathbf{u(x)} = y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Using x = 2, we get,

$$\mathbf{u(2)} = u_2 = -\frac{10}{7} + \frac{15}{2} + \frac{55}{4} + \frac{5}{28}$$

$\therefore u(2) = u_2 = 20$

Problem: 5

Using Lagrange's interpolation formula, fit a polynomial to the following data.

x :	0	1	3	4
y :	-12	0	6	12

Solution:

Given the data $x_0 = 0, \quad x_1 = 1, \quad x_2 = 3, \quad x_3 = 4$

$y_0 = -12, \quad y_1 = 0, \quad y_2 = 6, \quad y_3 = 12$

By Lagrange's interpolation,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = f(x) = \frac{(x-1)(x-3)(x-4)}{-12}(-12) + \frac{x(x-1)(x-4)}{(-6)}(6) + \frac{x(x-1)(x-3)}{12}(12)$$

$$\therefore f(x) = x^3 - 8x^2 + 19x - 12 - \left(x^3 - 5x^2 + 4x\right) + x^3 - 4x^2 + 3x$$

$\Rightarrow f(x) = x^3 - 7x^2 + 18x - 12$ is the required polynomial.

Home work Problems:

1. Using lagrange's interpolation, Calculate the profit in the year 2000 from the following data

Year :	1997	1999	2001	2002
Profit:	43	65	159	248

2. Find the missing term in the following table using lagrange's interpolation

X	0	1	2	3	4
y	1	3	9	-	81

3. Using Lagrange's interpolation formula, find the equation of the cubic curve passes through the points (-1,-8),(0,3),(2,1),and (3,2).

4. Fit the third degree polynomial f(x) and to find f(4) satisfying the following data Using Lagrange's interpolation formula

X	1	3	5	17
Y	24	120	336	720

Home work Problems:

Answers:

1. y(2000) =100

2. y(3) = 31

3. $y = \dfrac{7x^3 - 31x^2 + 28x + 18}{6}$

4. y(x) = $x^3 + 6x^2 + 11x + 6$ and f(4)=210

Newton's Divided Difference Formula: (For Unequal intervals)

Let $f(x_0)$, $f(x_1)$, …. $f(x_n)$ be the values of $f(x)$ corresponding to the arguments $x_0, x_1, \ldots. x_n$, not necessarily equally spaced. Then

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \ldots.$$

$$\ldots\ldots\ldots + (x - x_0)(x - x_1)\ldots\ldots\ldots(x - x_{n-1}) f(x_0, x_1, x_2, \ldots x_n)$$

This formula is called Newton's Divided Difference formula.

Representation by Divided difference table

Argument x	Entry $f(x)$	First Divided difference $\Delta_1 f(x)$	Second Divided difference $\Delta_1^2 f(x)$	Third Divided difference $\Delta_1^3 f(x)$	Fourth Divided difference $\Delta_1^4 f(x)$
x_0	$f(x_0)$				
		$\frac{f(x_1)-f(x_0)}{x_1-x_0}=f(x_0,x_1)$			
x_1	$f(x_1)$		$f(x_0,x_1,x_2)$		
		$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f(x_1,x_2)$		$f(x_0,x_1,x_2,x_3)$	
x_2	$f(x_2)$		$f(x_1,x_2,x_3)$		$f(x_0,x_1,x_2,x_3,x_4)$
		$\frac{f(x_3)-f(x_2)}{x_3-x_2}=f(x_3,x_2)$		$f(x_1,x_2,x_3,x_4)$	
x_3	$f(x_3)$		$f(x_2,x_3,x_4)$		
		$\frac{f(x_4)-f(x_3)}{x_4-x_3}=f(x_4,x_3)$			
x_4	$f(x_4)$				

Problem: 1

Using Newton's divided difference method, find the value of f(8) and f(15), given the following data.

x:	4	5	7	10	11	13
f(x):	48	100	294	900	1210	2028

Solution:

The divided difference table is given below

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$	$\Delta_1^4 f(x)$
4	48 $f(x_0)$				
		$f(x_0,x_1)$ $\frac{100-48}{5-4}=52$			
5	100		$f(x_0,x_1,x_2)$ $\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$f(x_0,x_1,x_2,x_3)$ $\frac{21-15}{10-4}=1$	
7	294		$\frac{202-97}{10-5}=21$		$f(x_0,x_1,x_2,x_3,x_4)$ 0
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
10	900		$\frac{310-202}{11-7}=27$		0
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
11	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

By Newton's Divided Difference interpolation formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_o, x_1, x_2, x_3) + \ldots\ldots \quad \textbf{(1)}$$

From the given data,

$$x_0 = 4, x_1 = 5, x_2 = 7, x_3 = 10, x_4 = 11, x_5 = 13 \; and \; f(x_0) = 48$$

Using the divided differences and the given data in (1),

$$f(x) = 48 + 52(x-4) + 15(x-4)(x-5) + (x-4)(x-5)(x-7)$$

When x = 8,

f(8) = 48 + 208 + 180 + 12

$\Rightarrow$ f(8) = 448

When x = 15,

f(15) = 48 + 572 + 1650 + 880

$\Rightarrow$ f(15) = 3150

Problem: 2

Use Newton's divided difference formula, to fit a polynomial to the data

x:	-1	0	2	3
y:	-8	3	1	12

and hence find y when $x = 1$

Solution:

The divided difference table is given below

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$
-1	-8 $f(x_0)$			
		$f(x_0,x_1)$ $\frac{11}{1}=11$		
0	3		$f(x_0,x_1,x_2)$ $\frac{-12}{3}=-4$	
		$\frac{-2}{2}=-1$		$f(x_0,x_1,x_2,x_3)$ $\frac{8}{4}=2$
2	1		$\frac{12}{3}=4$	
		$\frac{11}{1}=11$		
3	12			

By Newton's Divided Difference interpolation formula,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_o,x_1,x_2,x_3)+..... \quad \textbf{(1)}$$

Here $f(x_0)=-8\,, x_0=-1, x_1=0, x_2=2, x_3=3$

$f(x_0,x_1)=11\,, f(x_0,x_1,x_2)=-4 \; and \; f(x_0,x_1,x_2,x_3)=2$

Using these we get,

$$f(x)=-8+(x+1)11+(x+1)x(-4)+(x+1)x(x-2)2$$
$$=-8+11x+11-4x^2-4x+2x^3-2x^2-4x$$

$$\therefore y=2x^3-6x^2+3x+3$$

$$\Rightarrow y(1)=2-6+3+3=2$$

Problem: 3

Find the eqn. of the cubic curve which passes through the points (4, -43), (7, 83), (9, 327) and (12, 1053). Hence find f(10), using divided difference interpolation formula.

Solution: The divided difference table is given below

x	f(x)	$\Delta_1 f(x)$	$\Delta_1^{\,2} f(x)$	$\Delta_1^{\,3} f(x)$
4	$f(x_0)$ -43			
		$f(x_0,x_1)$ $\frac{83+43}{7-4} = 42$		
			$f(x_0,x_1,x_2)$ $\frac{122-42}{9-4} = 16$	
7	83			$f(x_0,x_1,x_2,x_3)$ $\frac{24-16}{12-4} = 1$
		$\frac{327-83}{9-7} = 122$		
9	327		$\frac{242-122}{412-7} = 24$	
		$\frac{1053-327}{12-9} = 242$		
12	1053			

By Newton's Divided Difference interpolation formula,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$

$$+(x-x_0)(x-x_1)(x-x_2)f(x_o,x_1,x_2,x_3)+\ldots\ldots \quad \textbf{(1)}$$

Given data $x_0 = 4,\ x_1 = 7,\ x_2 = 9,\ x_3 = 12$

$f(x_0) = -43,\ f(x_0, x_1) = 42,\ f(x_0, x_1, x_2) = 16,\ f(x_0, x_1, x_2, x_3) = 1$

Using these, we get,

$$f(x) = -43 + (x-4)(42) + (x-4)(x-7)(16) + (x-4)(x-7)(x-9)(1)$$

$$= -43 + 42x - 168 + (x^2 - 11x + 28)(16) + (x^2 - 11x + 28)(x-9)$$

$$f(x) = x^3 - 4x^2 - 7x - 15$$

Put x =10 in above eqn .we get

$$f(10) = 515$$

Problem: 4

Find the polynomial equation $y = f(x)$ passing through $(-1,3), (0,-6), (3,39), (6,822)$ and $(7,1611)$

Solution: Given data

x:	-1	0	3	6	7
y:	3	-6	39	822	1611

The divided difference table is given below

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$	$\Delta_1^4 f(x)$
-1	3 $f(x_0)$				
		$f(x_0,x_1)$ $\frac{-9}{1}=-9$			
0	-6		$f(x_0,x_1,x_2)$ $\frac{24}{4}=6$		
		$\frac{45}{3}=15$		$f(x_0,x_1,x_2,x_3)$ $\frac{35}{7}=5$	
3	39		$\frac{246}{6}=41$		$f(x_0,x_1,x_2,x_3,x_4)$ $\frac{8}{8}=1$
		$\frac{783}{3}=261$		$\frac{91}{7}=13$	
6	822		$\frac{528}{4}=132$		
		$\frac{789}{1}=789$			
7	1611				

By Newton's Divided Difference interpolation formula,

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2) f(x_o, x_1, x_2, x_3) + ... \quad \textbf{(1)}$$

Given data $x_0 = -1, x_1 = 0, \quad x_2 = 3, \quad x_3 = 6, x_4 = 7$

$f(x_0) = 3, f(x_0, x_1) = -9, f(x_0, x_1, x_2) = 6, f(x_0, x_1, x_2, x_3) = 5,$

$f(x_0, x_1, x_2, x_3, x_4) = 1$

$$y = f(x) = 3 + (x+1)(-9) + (x+1)x(6) + (x+1)x(x-3)5$$

$$+ (x+1)x(x-3)(x-6)1$$

$$\Rightarrow y = 3 - 9x - 9 + 6x^2 + 6x + 5x^3 - 10x^2 - 15x + x(x^3 - 8x^2 + 9x + 18)$$

$y = x^4 - 3x^3 + 5x^2 - 6$ is the required polynomial

Home work Problems:

1. Using Newton's divided difference method find f(1.5) using the data f(1.0)=0.7651977, f(1.3)=0.6200860, f(1.6)=0.4554022, f(1.9)=0.2818186 and f(2.2)=0.1103623.

2. Find f(1), f(5) and f(9) using Newton's divided difference formula from the following table:

x :	0	2	3	4	7	8
f(x) :	4	26	58	112	466	668

3. Using Newton's divided difference formula, find the value u(3) given u(1) = -26, u(2) = 12 , u(4) = 256 and u(6) = 844.

4. Find f(x) as a polynomial in x for the following data by Newton's divided difference formula:

x:	-4	-1	0	2	5
f(x):	1245	33	5	9	1335

Cubic Spline

Definition: Cubic Spline

A Cubic spline S(x) is defined by the following properties.

(i) $S(x_i) = y_i$, where i = 0,1,2,…n

(ii) $S(x)$, $S'(x)$, $S''(x)$ are continous on closed interval [a, b]

(iii) S(x) is atmost a cubic polynomial in each interval (x_{i-1}, x_i), i = 1,2,3,….n

Definition: Natural Cubic Spline

The natural or free conditions $\mathbf{S''(x_0) = M_0 = 0, S''(x_n) = M_n = 0}$ give the natural cubic spline.

Formula:

1). For equal intervals, $x_{i+1} - x_i = h$, we have the relation

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \qquad i = 1,2,3, \dots \dots \dots (n-1)$$

2).If S(x) is the cubic spline in $x_{i+1} \le x \le x_i$, then

$$s(x) = y(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}[x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right]$$
$$+ \frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right] \quad i = 1,2,3 \dots ..$$

Problem: 1

Fit a Cubic Spline for the following Data and hence evaluate y(1.5) .

x	1	2	3
y	-6	-1	16

Solution:

Given Data

	x_0	x_1	x_2
x	1	2	3
y	-6	-1	16
	y_0	y_1	y_2

Here h = 1.Assume that $M_0 = 0$ and $M_2 = 0$

To find M_1, use the following formula

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \qquad i = 1,2,3,\dots\dots\dots(n-1)$$

Put i =1 in above equation, we get

$$M_0 + 4\,M_1 + M_2 = \frac{6}{h^2}[y_0 - 2y_1 + y_2]$$

$$0 + 4M_1 + 0 = \frac{6}{1}[-6 - 2(-1) + 16]$$

$$4M_1 = 6[12]$$

$$M_1 = 18$$

The Cubic Spline In the interval $x_{i-1} \le x \le x_i$ *is given by*

$$s(x) = y(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}[x_i - x)\left[y_{i-1} - \frac{h^2}{6} M_{i-1}\right]$$

$$+ \frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6} M_i\right] \quad i = 1,2,3 \ldots\ldots n$$

For $1 \le x \le 2$, put i =1 in above eqn ,We get

$$s(x) = y(x) = \frac{1}{6}[(x_1 - x)^3 M_O + (x - x_0)^3 M_1] + \frac{1}{1}(x_1 - x)\left[y_0 - \frac{1}{6} M_0\right]$$

$$+ \frac{1}{1}(x - x_{0)}[y_1 - \frac{1}{6} M_1]$$

$$= \frac{1}{6}[0 + (x - 1)^3(18)] + (2 - x)[-6 - 0] + (x - 1)[-1 - 3]$$

$$= 3(x - 1)^3 - 12 + 6x - 4x + 4$$

$$=3[x^3 - 3x^2 + 3x - 1] + 2x - 8$$

$$= 3x^3 - 9x^2 + 9x - 3 + 2x - 8$$

$$y(x) = 3x^3 - 9x^2 + 11x - 11, \qquad 1 \le x \le 2$$

Since x = 1.5 lies in the interval $1 \le x \le 2$, Put x = 1.5 in

$y(x) = 3x^3 - 9x^2 + 11x - 11,$

We get, **y (1.5) = -4.6250**

Problem: 2

Using Cubic Spline, find y(0.5) and $y'(1)$ from the following data assuming that $y''(0)$ and $y''(2) = 0$

x	0	1	2
y	-5	-4	3

Solution:

Given Data

	x_0	x_1	x_2
x	0	1	2
y	-5	-4	3
	y_0	y_1	y_2

Here h = 1. Assume that $M_0=0$ and $M_2=0$

To Find M_1, Use the following formula

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \qquad i = 1,2,3, \dots \dots \dots (n-1)$$

Put i =1 in the above eqn., We get,

$$M_0 + 4\,M_1 + M_2 = \frac{6}{h^2}[y_0 - 2y_1 + y_2]$$

$$0 + 4M_1 + 0 = \frac{6}{1}[-5 - 2(-4) + 3]$$

$$4M_1 = 6[6] \Rightarrow M_1 = 9$$

The Cubic Spline in the interval $x_{i-1} \le x \le x_i$ is given by

$$s(x) = y(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}[x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right]$$

$$+\frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right] \qquad i = 1,2,3 \ldots ..$$

For $0 \le x \le 1$, put i =1 in the above eqn., We get,

$$y(x) = \frac{1}{6}[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1] + (x_1 - x)\left[y_0 - \frac{1}{6}M_0\right]$$

$$+(x - x_0)[y_1 - \frac{1}{6}M_1]$$

$$= \frac{1}{6}[(1 - x)^3.0 + (x - 0)^3(9)] + (1 - x)[-5 - 0]$$

$$+(x - 0)[-4 - \frac{1}{6}(9)]$$

$$= \frac{3}{2}x^3 - 5 + 5x + x\left(\frac{-11}{2}\right)$$

$$= \frac{3}{2}x^3 - \frac{x}{2} - 5$$

$$y(x) = \frac{1}{2}[3x^3 - x - 10] \qquad 0 \le x \le 1 \text{ -------------} \rightarrow (1)$$

$$y'(x) = \frac{1}{2}[9x^2 - 1] \text{ -----------------} \rightarrow (2)$$

Put x = 0.5 in eqn. (1), we get y(0.5) = -5.0625

Put x = 1 in eqn. (2), We get, $y'(1) = 4$

Problem: 3

From the following table, fit the Polynomial and Compute y(1.5) and $y^1(1)$ using Cubic Spline.

x	1	2	3
y	-8	-1	18

Solution:

Given Data

	x_0	x_1	x_2
x	1	2	3
y	-8	-1	18
	y_0	y_1	y_2

Here h = 1 .Assume that M_0=0 and M_2=0

To Find M_1, Use the following formula

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \qquad i = 1,2,3, \dots \dots \dots (n-1)$$

Put i =1 in the above equation, We get

$$M_0 + 4\,M_1 + M_2 = \frac{6}{h^2}[y_0 - 2y_1 + y_2]$$

$$0 + 4M_1 + 0 = 6[-8 - 2(-1) + 18]$$

$$4M_1 = 6[12]$$

$$M_1 = 18$$

The Cubic Spline in the interval $x_{i-1} \le x \le x_i$ is given by

$$s(x) = y(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}[x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right]$$
$$+ \frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right] \qquad i = 1,2,3 \dots ..$$

For $1 \le x \le 2$, put i =1 in the above eqn., We get,

$$y(x) = \frac{1}{6}[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1] + (x_1 - x)\left[y_0 - \frac{1}{6}M_0\right] + (x - x_0)[y_1 - \frac{1}{6}M_1]$$

$$y(x) = \frac{1}{6}[(2 - x)^3.0 + (x - 1)^3(18)] + (2 - x)[-8] + (x - 1)[-1 - 3]$$
$$= 3(x - 1)^3 - 16 + 8x - 4x + 4$$
$$= 3[x^3 - 3x^2 + 3x - 1] - 12 + 4x$$
$$= 3x^3 - 9x^2 + 9x - 3 - 12 + 4x$$

$$\mathbf{y(x) = 3x^3 - 9x^2 + 13x - 15} \quad \longrightarrow (1)$$

$$\mathbf{y'(x) = 9x^2 - 18x + 13} \quad \longrightarrow (2)$$

Put x =1.5 in equation (1), We get
y(1.5) =-5.6250

Put x =1 in equation(2), We get
$\mathbf{y'(1) = 4}$

Problem: 4

Fit a Cubic Splines for the following data.

x	1	2	3	4
y	1	2	5	11

Solution:

Given Data

	x_0	x_1	x_2	x_3
x	1	2	3	4
y	1	2	5	11
	y_0	y_1	y_2	y_3

Here h = 1. Assume that $M_0 = 0$ and $M_3 = 0$

To Find M_1 and M_2, Use the following formula

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}],\ i = 1,2,3,\ldots\ldots\ldots(n-1) \text{---}\rightarrow(1)$$

Put i =1 in eqn.(1), We get,

$$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$$

$$0 + 4\ M_1 + M_2 = 6[1 - 2(2) + 5]$$

$$4\ M_1 + M_2 = 12 \text{--------------------}\rightarrow \text{(A)}$$

Put i =2 in eqn.(1), we get,

$$M_1 + 4M_2 + M_3 = 6[y_1 - 2y_2 + y_3]$$

$$0 + \ M_1 + 4M_2 = 6[2 - 2(5) + 11]$$

$$M_1 + 4M_2 = 18 \text{--------------------}\rightarrow \text{(B)}$$

$(A) \Rightarrow \; 4M_1 + M_2 = 12$

$(B) \times 4 \Rightarrow \; 4M_1 + 16 = 72 \quad (-)$

$-15M_2 = -60$ Solving, We get, $M_2 = 4$

Put $M_2 = 4$ in (A), we get $4M_1 + 4 = 12 \Rightarrow M_1 = 2$

The Cubic Spline in the interval $x_{i-1} \le x \le x_i$ is given by

$$s(x) = y(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}[x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right]$$

$$+\frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right] \qquad i = 1,2,3 \ldots \ldots \text{-----------} \rightarrow (2)$$

For $1 \le x \le 2$, put i =1 in eqn.(2) We get, the cubic spline,

$$y(x) = \frac{1}{6}[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1] + (x_1 - x)\left[y_0 - \frac{1}{6}M_0\right]$$

$$+(x - x_0)[y_1 - \frac{1}{6}M_1]$$

$$= \frac{1}{6}[0 + (x-1)^3(2)] + (2-x)[1-0] + (x-1)[2 - \frac{1}{3}]$$

$$= \frac{1}{3}[x^3 - 3x^2 + 3x - 1] + 2 - x + \frac{5}{3}x - \frac{5}{3}$$

$$= \frac{1}{3}[x^3 - 3x^2 + 3x - 1] + \frac{1}{3} + \frac{2}{3}x$$

$$y(x) = \frac{1}{3}[x^3 - 3x^2 + 5x] \quad , 1 \le x \le 2$$

For $2 \le x \le 3$, put i =2 in eqn.(2) We get, the cubic spline,

$$y(x) = \frac{1}{6}[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2] + (x_2 - x)\left[y_1 - \frac{1}{6}M_1\right]$$

$$+(x - x_1)[y_2 - \frac{1}{6}M_2]$$

$$= \frac{1}{6}[(3 - x)^3(2) + (x-2)^3(4)] + (3-x)(2-\frac{1}{3}] + (x - 2)[5 - \frac{2}{3}]$$

$$= \frac{1}{3}[(27 - 27x + 9x^2 - x^{3)} + 2(x^3 - 6x^2 + 12x - 8)]$$

$$+(3 - x)\left(\frac{5}{3}\right) + (x - 2)\frac{13}{3}$$

$$= \frac{1}{3}[27 - 27x + 9x^2 - x^3 + 2x^3 - 12x^2 + 24x - 16$$

$$+15 - 5x + 13x - 26]$$

$$y(x) = \frac{1}{3}[x^3 - 3x^2 + 5x] \qquad 2 \le x \le 3$$

For $3 \le x \le 4$, put i =3 in eqn(2) We get the cubic spline,

$$y(x) = \frac{1}{6}[(x_3 - x)^3 M_2 + (x - x_2)^3 M_3] + (x_3 - x)\left[y_2 - \frac{1}{6}M_2\right]$$

$$+(x - x_2)[y_3 - \frac{1}{6}M_3]$$

$$= \frac{1}{6}[(4 - x)^3(4) + 0] + (4 - x)\left[5 - \frac{2}{3}\right] + (x - 3)[11 - 0]$$

$$= \frac{2}{3}[4^3 - 3(4)^2 x + 3(4)(x)^2 - x^3] + (4 - x)\left(\frac{13}{3}\right) + 11x - 33$$

$$= \frac{2}{3}[64 - 48x + 12x^2 - x^3] + \frac{52}{3} - \frac{13}{3}x + 11x - 33$$

$$= \frac{1}{3}[128 - 96x + 24x^2 - 2x^3] - \frac{47}{3} + \frac{20}{3}x$$

$$= \frac{1}{3}[128 - 96x + 24x^2 - 2x^3 - 47 + 20x]$$

$$y(x) = \frac{1}{3}[-2x^3 + 24x^2 - 76x + 81] \qquad 3 \le x \le 4.$$

Home work Problems:

1. Find the cubic splines from the table given below. Assume $M_0 = 0, M_3 = -12$.

x	0	2	4	6
y = f(x)	1	9	41	41

2. Find the cubic spline approximations for the function given below.

x	0	1	2	3
y = f(x)	1	2	33	244

Assume M(0) = M(3) = 0. Also find y(2.5).

Answers:

1. $M_1 = 12,\ M_2 = -12$

$y(x) = 1 + x^3, \quad 0 \le x \le 2$

$y(x) = 25 - 36x + 18x^2 - 2x^3, \quad 2 \le x \le 4$

$y(x) = -103 + 60x - 6x^2, \quad 4 \le x \le 6$

2. $M_1 = -24,\ M_2 = 276$

$y(x) = -4x^3 + 5x + 1, \quad 0 \le x \le 1$

$y(x) = 50x^3 - 162x^2 + 167x - 53, \quad 1 \le x \le 2$

$y(x) = -46x^3 + 414x^2 - 935x + 715, \quad 2 \le x \le 3$

$y(2.5) = 121.25$

INTERPOLATION

Interpolation is the process of finding the intermediate values of the function from a set of its values at specific points given in a tabulated form.

The following table represents a set of corresponding values of x and $y = f(x)$:

x :	x_0	x_1	x_2	x_3	. . . .	x_n
y :	y_0	y_1	y_2	y_3		y_n

The process of computing y corresponding to x where $x_i < x < x_{i+1}$, $i = 0,1,2, \ldots n-1$ is interpolation.

GREGORY-NEWTON'S FORWARD INTERPOLATION FORMULA FOR EQUAL INTERVALS

If $y_0, y_1, y_2, \dots y_n$ are the values of $y = f(x)$ corresponding to equidistant values of $x = x_0, x_1, x_2, \dots x_n$

where $x_i - x_{i-1} = h$, for $i = 0,1,2,\dots n$,

then

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots + \frac{u(u-1)\dots(u-\overline{n-1})}{n!}\Delta^n y_0$$

Where $u = \dfrac{x - x_0}{h}$

Forward difference table:

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

GREGORY -NEWTON'S BACKWARD INTERPOLATION FORMULA:
(for equal intervals)

If $y_0, y_1, y_2, \ldots y_n$ are the values of $y = f(x)$ corresponding to equidistant values of $x = x_0, x_1, x_2, \ldots x_n$

where $x_i - x_{i-1} = h$, for $i = 0,1,2,\ldots n$,

then

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

Where $v = \frac{(x - x_n)}{h}$

Problem: 1

Find the values of y at x=21 and x=28 from the following data:

x	20	23	26	29
y	0.3420	0.3907	0.4384	0.4848

Solution:

Newton's forward difference table

x	y = f(x)	Δ_y	$\Delta^2{}_y$	$\Delta^3{}_y$
x_0 20	y_0 0.3420			
		Δy_0 0.0487		
23	0.3907		$\Delta^2 y_0$ -0.0010	
		0.0477		$\Delta^3 y_0$ -0.0003
26	0.4384		-0.0013	$\nabla^3 y_n$
		0.0464	$\nabla^2 y_n$	
29 x_n	0.4848 y_n	∇y_n		

Since x=21 is nearer to the beginning of the table , we use Newton's forward formula

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Where $u = \frac{(x-x_0)}{h}$, h is the interval of differencing.

$$u = \frac{(x-x_0)}{h} = \frac{21-20}{3} = 0.3333$$

$$y(21) = 0.3420 + \frac{0.3333}{1!}(0.0487) + \frac{0.3333(0.3333-1)}{2!}(-0.001)$$

$$+\frac{0.3333(0.3333-1)(0.3333-2)}{3!}(-0.0003)$$

$$= 0.3420 + (0.3333)(0.0487) + \frac{0.3333(-0.6666)}{2}(-0.001)$$

$$+\frac{0.3333(-0.6666)(-1.6666)}{6}(-0.0003)$$

$$y(21) = 0.3583$$

Since x=28 is nearer to end value, we use Newton's backward formula

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

Where $v = \frac{(x-x_n)}{h}$, h is the interval of differencing.

$$v = \frac{(28-29)}{3} = -0.3333$$

$$y(x) = 0.4848 + \frac{(-0.3333)}{1!}(0.0464)$$

$$+ \frac{(-0.3333)(-0.3333+1)}{2!}(-0.0013)$$

$$+ \frac{(-0.3333)(-0.3333+1)(-0.3333+2)}{3!}(-0.0003)$$

$$=0.4848+(-0.3333)(0.0464)+\frac{(-0.3333)(0.6667)}{2}(-0.0013)$$

$$+\frac{(-0.3333)(0.6667)(1.6667)}{6}(-0.0003)$$

$$=0.4848-0.01546+0.0001444+0.0000185$$

$$y(28)=0.4695$$

Problem: 2

The following data are taken from the steam table. Find the pressure at temperature $t = 142°C$ and $175°C$

Temp. $°C$	140	150	160	170	180
Pressure Kgf/cm^2	3.685	4.854	6.302	8.076	10.225

Solution:

Newton's forward difference table

Temp. (t)	Pressure (p)	Δp	$\Delta^2 p$	$\Delta^3 p$	$\Delta^4 p$
x_0	y_0				
140	**3.685**	Δy_0			
		1.169	$\Delta^2 y_0$		
				$\Delta^3 y_0$	
150	4.854		**0.279**		
					$\Delta^4 y_0$
		1.448		**0.047**	
			0.326		**0.002**
160	6.302				
		1.774		**0.049**	
			0.375		
170	8.076				
		2.149			
180	**10.225**				

Since t = 142 is nearer to the beginning of the table, we use Newton's forward formula

$$y(t) = p_0 + \frac{u}{1!}\Delta p_0 + \frac{u(u-1)}{2!}\Delta^2 p_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 p_0 + \cdots$$

Where $u = \frac{(t-t_0)}{h} = \frac{142-140}{10} = \frac{1}{5} = 0.2$

$$y(t=142) = 3.865 + \frac{0.2}{1!}(1.169) + \frac{0.2(0.2-1)}{2!}(0.279)$$

$$+ \frac{0.2(0.2-1)(0.2-2)}{3!}(0.047)$$

$$+ \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{3!}(0.002)$$

=3.685+0.2338-0.02332+0.002256-0.0000672

$$y(t=142) = 3.898$$

Since t = 175 is nearer to the end value, we use Newton's backward formula

$$y(x) = p_n + \frac{v}{1!}\nabla p_n + \frac{v(v+1)}{2!}\nabla^2 p_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 p_n + \cdots$$

$$v = \frac{(t-t_n)}{h} = \frac{(175-180)}{10} = \frac{-1}{5} = -0.5$$

$$y(t=175) = 10.225 + \frac{(-0.5)}{1!}(2.149) + \frac{(-0.5)(-0.5+1)}{2!}(0.375)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(0.049)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(0.002)$$

$$= 10.225 + \frac{(-0.5)}{1!}(2.149) + \frac{(-0.5)(0.5)}{2}(0.375)$$

$$+ \frac{(-0.5)(0.5)(1.5)}{6}(0.049)$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)}{24}(0.002)$$

=10.225-1.0745-0.046875-0.0030625-0.000078125

=9.10048438

$$y(t = 175) = 9.1005$$

Problem: 3

Using Newton's forward interpolation formula, find the polynomial f(x) satisfying the following data hence find f(2).

x	0	5	10	15
f(x)	14	379	1444	3584

Solution:

Newton's forward difference table

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
0	**14**			
		Δy_0		
		365	$\Delta^2 y_0$	
			700	$\Delta^3 y_0$
5	379			**375**
		1065		
			1075	
10	1444			
		2140		
15	3584			

Newton's Forward interpolation formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Where $u = \frac{(x-x_0)}{h}$, h is the interval of differencing.

$$u = \frac{(x-x_0)}{h} = \frac{x-0}{5} = \frac{x}{5}$$

$$y = f(x) = 14 + \frac{x}{5}(365) + \frac{x}{5}\left(\frac{x}{5}-1\right)\left(\frac{700}{2}\right) + \left(\frac{x}{5}\right)\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)\left(\frac{375}{6}\right)$$

$$= 14 + 73x + \frac{x(x-5)}{25}(350) + \frac{x(x-5)(x-10)}{125 \times 6}(375)$$

$$= 14 + 73x + (x^2-5x)(14) + \frac{x(x^2-15x+50)}{2}$$

$$= \frac{1}{2}(28 + 146x + 28(x^2-5x) + x^3 - 15x^2 + 50x)$$

$$y = \frac{1}{2}(x^3 + 13x^2 + 56x + 28) \qquad y = f(2) = 100.$$