Differential Geometry

Unit - I

by

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Unit - I: Curves

Sec: 1.1 : Curves in space

A curve in space is the motion of a point under certain conditions. Here we use the rectangular Cartesian coordinates (x, y, z) or (x_1, x_2, x_3).

One method of representing the curve in the space is the parametric form. $X_i = f_i(u)$, i = 1, 2, 3, ... Where u is a parameter.

(i.e) $x_1 = f_1(u)$, $x_2 = f_2(u)$, $x_3 = f_3(u)$ (or) $x = f_1(u)$, $y = f_2(u)$, $z = f_3(u)$

Another way of representing a curve in space is the intersection of two curves $F_1(x, y, z) = 0$ and $F_2(x, y, z) = 0$ (or) $y = f_1(x)$, $z = f_2(x)$

The intersection of the sphere and the plane is a circle in space.

The equation of the sphere $F_1 = x^2 + y^2 + z^2 - 9 = 0$ and the plane $F_2 = 2x + 3y - 5z + 4 = 0$ represents a circle in space.

(e.g.) $x_i = a_i + ub_i$, i = 1, 2, 3,... represent a straight line in space.

This can be written as $\frac{x_1 - a_1}{b_1} = \frac{x_2 - a_2}{b_2} = \frac{x_3 - a_3}{b_3}$.

Arc Length and Tangent:

Suppose that the curve c, $\bar{x} = \bar{x}(u)$ is real with real parameter u. Then we can express the arc length of a segment of a curve between A(u_0) and P(u) by means of the integral

$$S(u) = \int_{A(u_0)}^{P(u)} \sqrt{\dot{\bar{x}} \cdot \dot{\bar{x}}}\, du \text{ where } \dot{\bar{x}} = \frac{d\bar{x}}{du}$$

$$= \int_{u_0}^{u} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, du$$

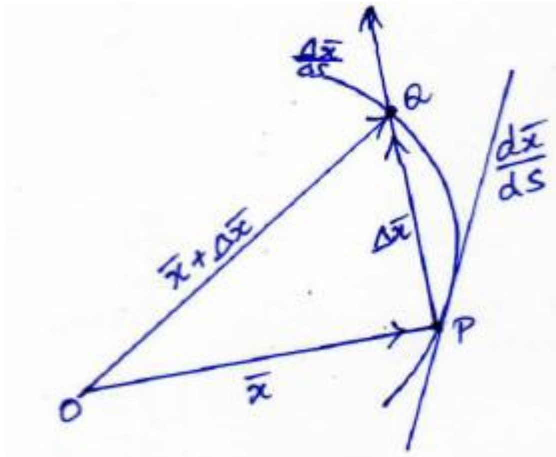
The arc length S is increasing with increasing u. The sense of increasing arc length is called the positive sense on the curve. A curve with a sense on it is called an oriented curve.

We have $ds^2 = dx^2 + dy^2 + dz^2$

$$= d\bar{x} . d\bar{x}$$

Now divide by ds^2, we get,

$1 = \frac{d\bar{x}}{ds} . \frac{d\bar{x}}{ds}$, $\quad \frac{d\bar{x}}{ds}$ is a unit vector.



The vector $\Delta\bar{x}$ joins two points P($\bar{x}$) and Q($\bar{x} + \Delta\bar{x}$) on the curve c. The vector $\frac{\Delta\bar{x}}{\Delta s}$ has the same direction as $\Delta\bar{x}$ and $\Delta s \to 0$. $\frac{\Delta\bar{x}}{\Delta s}$ gives tangent vector at the point p.

$\therefore \frac{d\bar{x}}{ds}$ gives the direction of tangent at p. Thus $\frac{d\bar{x}}{ds}$ is the unit tangent vector. It is denoted by $\bar{t}$.

Note: 1

$\frac{d\bar{x}}{ds} = \frac{d\bar{x}}{du}.\frac{du}{ds} = \dot{\bar{x}}.\frac{du}{ds}$ where $\dot{\bar{x}}$ is a tangent vector.

Note : 2

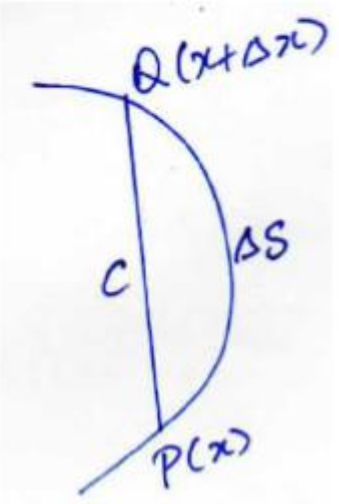
A tangent at a point on a curve is a straight line passing through two consecutive points on the curve.

Theorem: 1

The ratio of the arc and the chord connecting two points P and Q on a curve approaches unity when Q approaches P.

Proof:

Let Δs be the length of the arc PQ and C be the length of the chord PQ.



Then $PQ = C = \sqrt{(x+\Delta x-x)^2+(y+\Delta y-y)^2+(z+\Delta z-z)^2}$

$$= \sqrt{\Delta x^2+\ \Delta y^2+\Delta z^2}$$

$$\frac{\Delta s}{c} = \frac{\Delta s}{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}}$$

Divide both Numerator and Denominator by Δu on the RHS

$$\frac{\Delta s}{c} = \frac{\Delta s}{\Delta u}.\frac{1}{\sqrt{\left(\frac{\Delta x}{\Delta u}\right)^2+\left(\frac{\Delta y}{\Delta u}\right)^2+\left(\frac{\Delta z}{\Delta u}\right)^2}}$$

Taking limit on both sides, we have

$$\lim \frac{\Delta s}{c} = \lim \frac{\Delta s}{\Delta u} \cdot \frac{1}{\sqrt{\left(\frac{\Delta x}{\Delta u}\right)^2 + \left(\frac{\Delta y}{\Delta u}\right)^2 + \left(\frac{\Delta z}{\Delta u}\right)^2}}$$

$$= \frac{\dot{s}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}$$

$$= \frac{\dot{s}}{\sqrt{\dot{s}^2}} = \frac{\dot{s}}{\dot{s}} = 1$$

Hence the proof.

Note:

We have the unit tangent vectors as $\bar{t} = Cos\alpha_1 \overline{e_1} + Cos\alpha_2 \overline{e_2} + Cos\alpha_3 \overline{e_3}$ where $\overline{e_1}, \overline{e_2}, \overline{e_3}$ are unit vectors along the coordinate axis, $Cos\alpha_1, Cos\alpha_2, Cos\alpha_3$ are the cosine of the angle made by the tangent vector with the positive direction of the x axis.

Equation of Tangent:

A generic point $A(\overline{X})$ on tangent line at P is determined by the equation $\overline{x} = \overline{x} + v\overline{t}$ where $v = PA$.

The equation of the tangent line is given by $\frac{X_1 - x_1}{\frac{dx_1}{ds}} = \frac{X_2 - x_2}{\frac{dx_2}{ds}} = \frac{X_3 - x_3}{\frac{dx_3}{ds}}$

(i.e.) $\frac{X_1 - x_1}{Cos\alpha_1} = \frac{X_2 - x_2}{Cos\alpha_2} = \frac{X_3 - x_3}{Cos\alpha_3}$

Problem : 1

Find the equation of the tangent line to the circle x = a cosu, y = a sinu and z = o.

Solution:

The equation of the circle is given as x = a cosu, y = a sinu and z = o with centre at 0 and radius a.

$$\frac{dx}{du} = \dot{x} = -a\ sinu, \qquad \frac{dy}{du} = \dot{y} = a\ Cosu\ and\ \frac{dz}{du} = \dot{z} = 0$$

$$\left(\frac{ds}{du}\right)^2 = \left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$$

$$= (-a\ \sin u)^2 + (a\ \cos u)^2 + 0$$

$$= a^2\sin^2 u + a^2\cos^2 u = a^2$$

$$\left(\frac{ds}{du}\right) = a$$

$$ds = a\ du$$

Integrating, we get, $\int ds = \int a du$

$$s = au + c$$

Take c = 0, we get, s = au

$$\Rightarrow u = \frac{s}{a} \quad \Rightarrow \frac{du}{ds} = \frac{1}{a}$$

Now the unit tangent vector at any point on the curve is given by

$$\bar{t} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

$$= \left(\frac{dx}{du}.\frac{du}{ds}, \frac{dy}{du}.\frac{du}{ds}, \frac{dz}{du}.\frac{du}{ds}\right) = \left(-a\sin u.\frac{1}{a}, a\cos u.\frac{1}{a}, 0.\frac{1}{a}\right)$$

$$= (-\sin u, \cos u, o)$$

The equation of tangent is given by

$$\frac{X-x}{dx/_{ds}} = \frac{Y-y}{dy/_{ds}} = \frac{Z-z}{dz/_{ds}}$$

$$\frac{X-acosu}{-sinu} = \frac{Y-asinu}{cosu} = \frac{Z-0}{0}$$

(i.e.) $$\frac{X-acosu}{-sinu} = \frac{Y-asinu}{cosu}$$

Problem : 2

Find the equation of the tangent to the circular helix.

Proof:

The equation of the circular helix is given as x = a cosu, y = a sinu and z = bu where a and b are constants.

$$\frac{dx}{du} = \dot{x} = -a\,sinu, \qquad \frac{dy}{du} = \dot{y} = a\,Cosu \text{ and } \frac{dz}{du} = \dot{z} = b$$

$$\left(\frac{ds}{du}\right)^2 = \left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$$

$$\left(\frac{ds}{du}\right)^2 = (-a\ \sin u)^2 + (a\ \cos u)^2 + b^2$$

$$= a^2\sin^2 u + a^2\cos^2 u + b^2$$

$$= a^2 + b^2$$

$$= c^2 \quad \text{where } c^2 = a^2 + b^2$$

$$\left(\frac{ds}{du}\right) = c$$

$$ds = c\ du$$

Integrating, we get, $\int ds = \int c\,du$

$$s = cu + d$$

Take $d = 0$, we get, $s = cu$

$$\Rightarrow u = \frac{s}{c} \quad \Rightarrow \frac{du}{ds} = \frac{1}{c}$$

Now the unit tangent vector at any point on the curve is given by

$$\bar{t} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

$$= \left(\frac{dx}{du}.\frac{du}{ds}, \frac{dy}{du}.\frac{du}{ds}, \frac{dz}{du}.\frac{du}{ds}\right)$$

$$= \left(-a\sin u.\frac{1}{a}, a\cos u.\frac{1}{a}, b.\frac{1}{c}\right) \quad = \left(\frac{-a}{c}\sin u, \frac{a}{c}\cos u, \frac{b}{c}\right)$$

The equation of tangent is given by

$$\frac{X-x}{dx/_{ds}}=\frac{Y-y}{dy/_{ds}}=\frac{Z-z}{dz/_{ds}}$$

$$\frac{X-acosu}{\frac{-a}{c}sinu}=\frac{Y-asinu}{\frac{a}{c}cosu}=\frac{Z-bu}{\frac{b}{c}}$$

(i.e.)

$$\frac{X-acosu}{-asinu}=\frac{Y-asinu}{acosu}=\frac{Z-bu}{b}$$

Definition: Surface of degree K

The surface is of degree k if it is intersected by a line in k – points.

Definition: Space curve of degree k

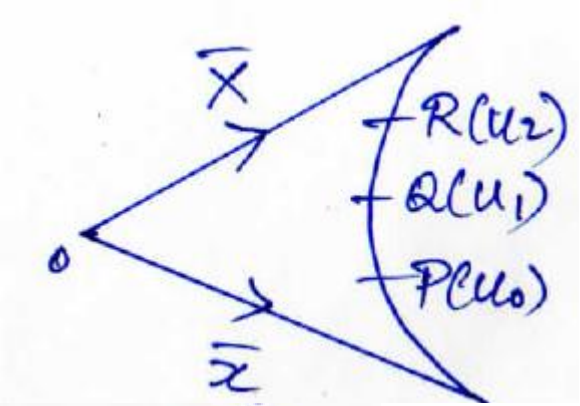
The space curve is said to be of degree k if it is intersected by a plane in k – points.

Sec: 1.3

Definition: Osculating plane

The osculating plane is a plane at a point on a space curve which is a limiting position of 3 consecutive points on the space curve, when two of the points coincides with the third.

Derivation of the equation of osculating plane



If $\overline{X}$ is any point on the plane passing through 3 consecutive points P(u_0), Q(u_1), R(u_2), then the equation of the plane is given by $\overline{X}.\overline{a} = p$ where $\overline{a}$ is perpendicular to the plane and p is a constant.

Let the plane be passing through three points P, Q, R on the curve given by

$$\overline{X} = \overline{x}(u_0), \overline{X} = \overline{x}(u_1), \overline{X} = \overline{x}(u_2)$$

then the function f(u) = $\overline{x}.\overline{a} - p$ where $\overline{X} = \overline{x}(u)$ satisfies the condition f(u_0) = 0, f(u_1) = 0, f(u_2) = 0.

According to Roll's theorem, we get $f'(u_3) = 0$, $f'(u_4) = 0$, $u_0 \le u_3 \le u_1$, $u_1 \le u_4 \le u_2$

Again applying Roll's theorem, we get $f''(u_5) = 0$, $u_3 \le u_5 \le u_4$

when Q and R approaches P, we get, $u_1, u_2, u_3, u_4, u_5 \to u_0$.

Taking $u_0 = u$, we obtain, for the limiting values of $\bar{a}$ and p, the condition $f(u) = 0$, $f'(u) = 0$ and $f''(u) = 0$.

(i.e) $\bar{x}.\bar{a} - p = 0,\ \dot{\bar{x}}.\bar{a} = 0, \ddot{\bar{x}}.\bar{a} = 0.$

Eliminating $\bar{a}$ from the above equation, we obtain a linear relationship among $\bar{X} - \bar{x},\ \dot{\bar{x}},\ \ddot{\bar{x}}$.

(i.e) $\bar{X} - \bar{x} = \lambda\dot{\bar{x}} + \mu\ddot{\bar{x}}$

$\bar{X} = \bar{x} + \lambda\dot{\bar{x}} + \mu\ddot{\bar{x}}$ where λ and μ are constants.

We can write $\bar{X} = \bar{x} + \lambda\dot{\bar{x}} + \mu\ddot{\bar{x}}$ in determinant form

$$\begin{vmatrix} X_1 - x_1 & X_2 - x_2 & X_3 - x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{vmatrix} = 0$$

Since $X_1 - x_1, \dot{\bar{x}}, \ddot{\bar{x}}$ are coplanar, we have $[\bar{X} - \bar{x},\ \dot{\bar{x}},\ \ddot{\bar{x}}.] = 0$.

This is the equation of the plane passing through 3 consecutive points. This equation is known as osculating plane.

Note : 1

The equation of the osculating plane passes through 3 consecutive points on the space curve.

Note : 2

It also passes through a tangent line given by $\bar{X} = \bar{x} + \lambda\dot{\bar{x}}$.

Note ; 3

The osculating plane is not determined when $\ddot{\bar{x}} = 0$ or $\ddot{\bar{x}}$ is proportional to $\dot{\bar{x}}$

Problem:3

Find the equation of osculating plane to the circular helix at any point.

Proof:

The equation of the circular helix is given as x = a cosu, y = a sinu and z = bu where a and b are constants.

$$\frac{dx}{du} = \dot{x} = -a\,sinu,\ \frac{dy}{du} = \dot{y} = a\,Cosu\ and\ \frac{dz}{du} = \dot{z} = b$$

$$\ddot{x} = -a\,Cosu, \ddot{y} = -a\,Sinu, \ddot{z} = 0.$$

Equation of osculating plane is given by

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

$$\begin{vmatrix} X-acosu & Y-asinu & Z-bu \\ -asinu & acosu & b \\ -acosu & -asinu & o \end{vmatrix} = 0$$

$$(X - acosu)(0 + asinu.b) - (Y - asinu)(0 + acosu.b) + (Z - bu)(a^2sin^2u + a^2cos^2u) = 0$$

$$X.\ asinu.\ b - a^2sinucosu.b - Yacosu.b + a^2sinucosu.b + za^2 - bua^2 = 0$$

$$a[bXsinu - bYcosu + az - abu] = 0$$

(i.e.) $bXsinu - bYcosu + az = abu$

Sec: 1.4 Curvature

Definition: Principal Normal

The line in the osculating plane at a point perpendicular to the tangent to the space curve is called principal normal. The unit vector along the principal normal is denoted by $\overline{n}$.

Derivation of curvature

Let the given curve be c represented by $\bar{x} = \bar{x}(s)$, where s is the arc length. Then $\bar{t} = \frac{d\bar{x}}{ds}$ is the unit tangent vector.

We have $\bar{t}.\bar{t} = 1$

Diff. w.r.t. s, we have

$$\bar{t}.\bar{t}' + \bar{t}'.\bar{t} = 0 \Rightarrow 2\bar{t}'.\bar{t} = 0 \Rightarrow \bar{t}'.\bar{t} = 0$$

(i.e) $\bar{t}'$ is perpendicular to $\bar{t}$.

Now $\bar{t} = \frac{d\bar{x}}{ds} = \frac{d\bar{x}}{du}.\frac{du}{ds} = \dot{\bar{x}}.\frac{du}{ds}$

Diff. this equation again w.r.t. s, we have

$$\bar{t}' = \frac{d}{ds}\left\{\dot{\bar{x}}\frac{du}{ds}\right\}$$

$$= \frac{d}{ds}(\dot{\bar{x}})\frac{du}{ds} + \dot{\bar{x}}\frac{d}{ds}\left(\frac{du}{ds}\right)$$

$$= \left(\frac{d\dot{\bar{x}}}{du}.\frac{du}{ds}\right)\frac{du}{ds} + \dot{\bar{x}}\frac{d^2u}{ds^2}$$

$$= \ddot{\bar{x}}\left(\frac{du}{ds}\right)^2 + \dot{\bar{x}}\frac{d^2u}{ds^2}$$

$\bar{t}'$ is the linear combination of $\dot{\bar{x}}$ and $\ddot{\bar{x}}$. Therefore $\bar{t}'$ lies in the osculating plane. Hence $\bar{t}^1$ and principal normal vector $\bar{n}$ are parallel.

$\therefore\ \bar{t}' = k\bar{n}$, where k is scalar.

If we denote $\bar{t}'$ as $\bar{k}$, then

$\bar{k} = k\,.\bar{n}$ where $\bar{k}$ is called the curvature vector.

The curvature vector expresses the rate of change of tangent, when we proceed along the curve.

$|\bar{k}|$ = k is called the length of the curvature vector.

Definition: Osculating Circle

It is a circle passing through three consecutive points of the space curve whose radius is the absolute value of the radius of curvature of the curve.

Note: $k^2 = \bar{k}\ .\bar{k} = \bar{t}'\ .\bar{t}' = \bar{x}''\ .\bar{x}''$

Here the radius of curvature is given by $R = k^{-1} = \frac{1}{k}$

Theorem: 1.4.1 (Derivative of the osculating circle)

The centre of the osculating circle lies on the principal normal at a distance $|R|$ and R is the radius of the osculating circle.

Proof:

We observe that the osculating circle lies in the osculating plane. The circle can be determined as the plane of intersection of osculating plane and a sphere.

The equation of the sphere is given by $(\overline{x}-\overline{c}).(\overline{x}-\overline{c}) = r^2$, where $\overline{x}$ is the generic point, of the sphere, $\overline{c}$ is the centre and $\overline{r}$ is radius.

This sphere must passes through three points, P, Q, R on the curve given by

$\overline{x} = \overline{x}\,(s_0), \overline{x} = \overline{x}\,(s_1), \overline{x} = \overline{x}\,(s_2)$

The vector $\overline{x}-\overline{c}$ lies in the osculating plane.

Let $f(s) = (\overline{x}-\overline{c}).(\overline{x}-\overline{c}) - r^2$.

Then for limiting values of $\overline{c}$ and r, we must here

$f(s) = 0, f'(s) = 0, f''(s) = 0$

$f(s) = 0 \Rightarrow (\overline{x}-\overline{c}).(\overline{x}-\overline{c}) - r^2 = 0 \quad \longrightarrow [1]$

$$f'(s) = 0 \Rightarrow \left(\overline{x}' - 0\right)(\overline{x} - \overline{c}) + (\overline{x} - \overline{c})\left(\overline{x}' - 0\right) = 0$$

$$\overline{x}'(\overline{x} - \overline{c}) + \overline{x}'(\overline{x} - \overline{c}) = 0$$

$$2\,\overline{x}'(\overline{x} - \overline{c}) = 0$$

$$\overline{x}'(\overline{x} - \overline{c}) = 0 \qquad \longrightarrow [2]$$

$$f''(s) = 0 \Rightarrow \overline{x}''(\overline{x} - \overline{c}) + \overline{x}'.\overline{x}' = 0$$

$$\overline{x}''(\overline{x} - \overline{c}) + \overline{t}.\overline{t} = 0$$

$$\overline{x}''(\overline{x} - \overline{c}) + 1 = 0$$

$$\overline{x}''(\overline{x} - \overline{c}) = -1 \qquad \longrightarrow [3]$$

Since $(\overline{x} - \overline{c})$ lies in the osculating plane,

We can write $(\overline{x} - \overline{c}) = \lambda\overline{x}' + \mu\overline{x}''$ $\longrightarrow [4]$

where λ and μ are constants which are determined by the equations [1], [2] & [3].

Take dot product with $\overline{x}'$ in equation [4], we have

$$(\overline{x}-\overline{c})\overline{x}' = \lambda\overline{x}'\overline{x}' + \mu\overline{x}''.x'$$

$$0 = \lambda\ \overline{t}\ .\overline{t} + \mu\,\overline{t}'.\overline{t}\ \text{(by equation [2])}$$

$$0 = \lambda\ (1) + \mu(0)\ \Rightarrow \lambda = 0$$

Take dot product with $\overline{x}''$ in equation [4], we have

$$(\overline{x}-\overline{c})\overline{x}'' = \lambda\overline{x}'\overline{x}'' + \mu\overline{x}''.\overline{x}''$$

$$-1 = \lambda\overline{t}\,\overline{t}' + \mu k^2\ \text{(by equation [3])}$$

$$-1 = \lambda(0) + \mu\, k^2\ \Rightarrow \mu = \frac{-1}{k^2}$$

$$\therefore [4] \Rightarrow (\overline{x}-\overline{c}) = \frac{-1}{k^2}\overline{x}''$$

$$\overline{c} = \overline{x} + \frac{1}{k^2}\overline{x}''$$

$$\overline{c} = \overline{x} + R\,\overline{n}\ \text{ where } \frac{\overline{x}''}{k} = \overline{n}\ \&\ \frac{1}{k} = \text{R.}$$

Sec: 1.5 Torsion

The curvature measures rate of change of tangent when moving along the curve. We shall now introduce a quantity, measuring the rate of change of osculating plane.

For this purpose, we introduce a normal at any point to the osculating. It is known as binormal. It is denoted by $\bar{b}$.

We may define $\bar{b}$ as $\bar{b} = \bar{t} \times \bar{n}$.

Now $\bar{t}$, $\bar{n}$, $\bar{b}$ satisfy the following relations

$$\bar{t}\,.\,\bar{t} = 1 = \bar{n}.\bar{n} = \bar{b}.\bar{b}$$

$$\bar{t}.\bar{x} = 0 = \bar{n}.\bar{b} = \bar{b}.\bar{t}$$

$$\bar{t} \times \bar{t} = 0 = \bar{n} \times \bar{n} = \bar{b} \times \bar{b}$$

$$\bar{t} \times \bar{n} = \bar{b}\,,\, \bar{n} \times \bar{b} = \bar{t},\; \bar{b} \times \bar{t} = \bar{n}$$

The rate of change of osculating plane is represented by $\bar{b}^1 = \dfrac{d\bar{b}}{ds}$

This vector lies in the direction of the principal normal.

Derivation of Torsion

We know that $\bar{b}.\bar{t} = 0$

Diff with respect to 's', we get

$$\bar{b}^{1}.\bar{t} + \bar{b}.\bar{t}^{1} = 0$$

$$\bar{b}^{1}.\bar{t} + \bar{b}.\,k\bar{n} = 0$$

$$\Rightarrow \bar{b}^{1}.\bar{t} = 0 \qquad [\because\ \bar{n}.\bar{b} = 0$$

(i.e.) $\bar{b}^{1}$ is perpendicular to $\bar{t}$ $\longrightarrow$ [1]

Now $\bar{b}.\bar{b} = 1$

Diff with respect to 's' we get

$$\bar{b}^{1}.\bar{b} + \bar{b}.\bar{b}^{1} = 0$$

$$2\,\bar{b}^{1}.\bar{b} = 0$$

(i. e) $\bar{b}^{1}$ is perpendicular to $\bar{b}$. --------(2)

from (1) & (2), we have $\bar{b}^{1}$ is perpendicular to both $\bar{t}$ and $\bar{b}$.

$\therefore \bar{b}^{1}$ lies in the direction of $\bar{n}$.

We can write $\bar{b}^{1} = -\tau\bar{n}$, for some suitable scalar, we call this τ as torsion of the curve.

Note:

τ may be positive or negative just like a curvature.

Derivation of expression for τ

Consider $\tau = (-\bar{n})\,(-\tau\bar{n})$

$$= (-\bar{n})\left(\frac{d\bar{b}}{ds}\right)$$

$$= (-\bar{n})\frac{d(\bar{t}\times\bar{n})}{ds}$$

$$= \left(\frac{-\bar{k}}{k}\right).\frac{d}{ds}\left(\bar{x}'\times\frac{-\bar{k}}{k}\right)$$

$$= \frac{-1}{k^2}\{\bar{k}(\bar{x}'\times\bar{x}'')'\}$$

$$= \frac{-1}{k^2}\{\bar{x}''(\bar{x}'\times\bar{x}'')'\} \quad ----[1]$$

$\because \bar{t} = \bar{x}'$

$\bar{k} = \bar{t}^1 = \bar{x}''$

$= k.\bar{n}$

$\Rightarrow \bar{n} = \frac{-\bar{k}}{k}$

Now $\bar{x}''(\bar{x}'\times\bar{x}'')' = \bar{x}''\{\bar{x}''\times\bar{x}'' + \bar{x}'\times\bar{x}'''\}$

$$= \bar{x}''\{0 + \bar{x}'\times\bar{x}'''\}$$

$$= \{\bar{x}'', \bar{x}', \bar{x}'''\}$$

$$= -[\bar{x}', \bar{x}'', \bar{x}''']$$

sub. this in (1), we get $\tau = \frac{\bar{x}', \bar{x}'', \bar{x}'''}{k^2}$ *or* $\tau = \frac{\bar{x}', \bar{x}'', \bar{x}'''}{\bar{x}''\,.\,\bar{x}''}$

Expression for curvature of torsion in terms of the Parameter U is

$$k^2 = \frac{(\dot{\bar{x}}\times\ddot{\bar{x}})).(\dot{\bar{x}}\times\ddot{\bar{x}})}{(\dot{\bar{x}}\times\ddot{\bar{x}})^3}$$

$$\tau = \frac{\dot{\bar{x}}, \ddot{\bar{x}}, \dddot{\bar{x}}}{(\dot{\bar{x}} \times \ddot{\bar{x}}).(\dot{\bar{x}} \times \ddot{\bar{x}})}$$

Problem: 1

Find the curvature and torsion for the circular helix.

Solution:

Equation of the Circular helix is $\dot{\bar{x}} = (acosu, asinu, bu)$

Take u= s/c where $c = \sqrt{a^2 + b^2}$

$$\therefore \bar{x} = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b.\frac{s}{c}\right)$$

$$\bar{x}' = \left[-a\sin\left(\frac{s}{c}\right).\left(\frac{1}{c}\right), a\cos\left(\frac{s}{c}\right).\left(\frac{1}{c}\right), \frac{b}{c}\right]$$

$$= \left[-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right]$$

$$\bar{x}'' = \left[-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right]$$

$$\bar{x}''' = \left[\frac{a}{c^3}\sin\left(\frac{s}{c}\right), -\frac{a}{c^3}\cos\left(\frac{s}{c}\right), 0\right]$$

Now , $k^2 = \bar{x}''.\bar{x}''$

$$= \frac{a^2}{c^4} cos^2\left(\frac{s}{c}\right) + \frac{a^2}{c^4} sin^2\left(\frac{s}{c}\right)$$

$$= \frac{a^2}{c^4}\left[cos^2\left(\frac{s}{c}\right) + sin^2\left(\frac{s}{c}\right)\right]$$

$$= \frac{a^2}{c^4}$$

$$k = \pm\sqrt{\frac{a^2}{c^4}} = \pm\frac{a}{c^2}$$

Now $[\bar{x}', \bar{x}'', \bar{x}'''] = \begin{vmatrix} -\frac{a}{c}\sin\left(\frac{s}{c}\right) & \frac{a}{c}\cos\left(\frac{s}{c}\right) & \frac{b}{c} \\ -\frac{a}{c^2}\cos\left(\frac{s}{c}\right) & -\frac{a}{c^2}\sin\left(\frac{s}{c}\right) & 0 \\ \frac{a}{c^3}\sin\left(\frac{s}{c}\right) & -\frac{a}{c^3}\cos\left(\frac{s}{c}\right) & 0 \end{vmatrix}$

$$= \frac{b}{c}\left[\frac{a^2}{c^5} cos^2\left(\frac{s}{c}\right) + \frac{a^2}{c^5} sin^2\left(\frac{s}{c}\right)\right]$$

$$= \frac{b}{c}.\frac{a^2}{c^5}\left[cos^2\left(\frac{s}{c}\right) + sin^2\left(\frac{s}{c}\right)\right]$$

$$= \frac{ba^2}{c^6}$$

$$\tau = \left[\frac{\bar{x}', \bar{x}'', \bar{x}'''}{\bar{x}''.\bar{x}''}\right] = \frac{\frac{ba^2}{c^6}}{\frac{a^2}{c^4}} = \frac{a^2 b}{c^6}.\frac{c^4}{a^2} = \frac{b}{c^2}$$

∴ Torsion at any point for the circular helix is constant.

Note:

At all points of the circular helix the curvature and torsion are constant.

Problem: 2

Find the curvature & torsion for the curve $x = u,\ \ y = u^2,\ z = u^3$

Solution:

Given $\bar{x} = (u, u^2, u^3)$

$$\dot{\bar{x}} = (1, 2u, 3u^2)$$

$$\ddot{\bar{x}} = (0, 2, 6u)$$

$$\dddot{\bar{x}} = (0, 0, 6)$$

$$\dot{\bar{x}} \times \ddot{\bar{x}} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix}$$

$$= \bar{\imath}[12u^2 - 6u^2] - \bar{\jmath}[6u - 0] + \bar{k}[2 - 0]$$

$$= 6u^2\bar{\imath} - 6u\bar{\jmath} + 2\bar{k}$$

$$(\dot{\bar{x}} \times \ddot{\bar{x}}).(\dot{\bar{x}} \times \ddot{\bar{x}}) = (6u^2\bar{\imath} - 6u\bar{\jmath} + 2\bar{k})(6u^2\bar{\imath} - 6u\bar{\jmath} + 2\bar{k})$$

$$= 36u^4 + 36u^2 + 4$$

$$= 4(9u^4 + 9u^2 + 1)$$

$$(\dot{\bar{x}}.\dot{\bar{x}}) = (1, 2u, 3u^2)(1, 2u, 3u^2)$$

$$= 1 + 4u^2 + 9u^4$$

$$(\dot{\bar{x}}.\dot{\bar{x}})^3 = (9u^4 + 4u^2 + 1)^3$$

Curvature is $k^2 = \frac{(\dot{\bar{x}}\times\ddot{\bar{x}})).(\dot{\bar{x}}\times\ddot{\bar{x}})}{(\dot{\bar{x}}\times\ddot{\bar{x}})^3}$

$$= \frac{4(9u^4+9u^2+1)}{(9u^4+4u^2+1)^3}$$

$$k = \pm 2\sqrt{\frac{9u^4+9u^2+1}{(9u^4+4u^2+1)^3}}$$

$$[\dot{\bar{x}}, \ddot{\bar{x}}, \dddot{\bar{x}}] = \begin{vmatrix} 1 & 2u & 3u^2 \\ 0 & 2 & 6u \\ 0 & 0 & 6 \end{vmatrix}$$

$$= 1(12 - 0) - 2u(0) + 3u^2(0) = 12$$

$$\therefore \tau = \frac{\dot{\bar{x}}, \ \ddot{\bar{x}}, \ \dddot{\bar{x}}}{(\dot{\bar{x}} \times \dot{\bar{x}}).(\dot{\bar{x}} \times \dot{\bar{x}})}$$

$$= \frac{12}{4(9u^4+9u^2+1)} = \frac{3}{(9u^4+9u^2+1)}$$

Problem: 3

Find the curvature & torsion for the curve $x = u,\ \ y = \frac{1+u}{u},\ \ z = \frac{1-u^2}{u}$

Solution:

Given $\ x = u,\ \ y = \frac{1+u}{u},\ \ z = \frac{1-u^2}{u}$

(i.e.) $x = u,\ y = \frac{1}{u} + 1,\ z = \frac{1}{u} - u$

$$\bar{x} = \left(u,\ \frac{1}{u} + 1,\ \frac{1}{u} - 1\right)$$

$$\dot{\bar{x}} = \left(1, -\frac{1}{u^2}, -\frac{1}{u^2} - 1\right)$$

$$\ddot{\bar{x}} = \left(0, \frac{2}{u^3}, \frac{2}{u^3}\right)$$

$$\dddot{\bar{x}} = \left(0, \frac{-6}{u^4}, \frac{-6}{u^4}\right)$$

$$\dot{\bar{x}} \times \ddot{\bar{x}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -\frac{1}{u^2} & -\frac{1}{u^2} - 1 \\ 0 & \frac{2}{u^3} & \frac{2}{u^3} \end{vmatrix}$$

$$= \vec{i}\left[-\frac{2}{u^5} + \frac{2}{u^5} + \frac{2}{u^3}\right] - \vec{j}\left[\frac{2}{u^3}\right] + \vec{k}\left[\frac{2}{u^3}\right]$$

$$\dot{\bar{x}} \times \ddot{\bar{x}} = \frac{2}{u^3}\left[\vec{i} - \vec{j} + \vec{k}\right]$$

$$(\dot{\boldsymbol{x}} \times \ddot{\boldsymbol{x}}).(\dot{\boldsymbol{x}} \times \ddot{\boldsymbol{x}}) = \left(\frac{2}{u^3}\vec{i} - \frac{2}{u^3}\vec{j} + \frac{2}{u^3}\vec{k}\right)\left(\frac{2}{u^3}\vec{i} - \frac{2}{u^3}\vec{j} + \frac{2}{u^3}\vec{k}\right)$$

$$= \frac{4}{u^6} + \frac{4}{u^6} + \frac{4}{u^6} = \frac{12}{u^6}$$

$$(\dot{\boldsymbol{x}} . \dot{\boldsymbol{x}}) = \left(1, -\frac{1}{u^2}, -\frac{1}{u^2} - 1\right)\left(1, -\frac{1}{u^2}, -\frac{1}{u^2} - 1\right)$$

$$= 1 + \frac{1}{u^4} + \frac{1}{u^4} + 1 + \frac{2}{u^2}$$

$$= \frac{2u^4+2+2u^2}{u^4} = \frac{2(u^4+u^2+1)}{u^4}$$

$$(\dot{\boldsymbol{x}} . \dot{\boldsymbol{x}})^3 = \frac{8(u^4+u^2+1)^3}{u^{12}}$$

$$\text{Curvature } k^2 = \frac{(\dot{\boldsymbol{x}}\times\ddot{\boldsymbol{x}})(\dot{\boldsymbol{x}}\times\ddot{\boldsymbol{x}})}{(\dot{\boldsymbol{x}} . \dot{\boldsymbol{x}})^3} = \frac{\left(\frac{12}{u^6}\right)}{\frac{8(u^4+u^2+1)^3}{u^{12}}}$$

$$= \frac{12}{u^6} \times \frac{u^{12}}{8(u^4+u^2+1)^3} = \frac{3}{2}\frac{u^6}{(u^4+u^2+1)^3}$$

$$\Rightarrow k = \pm\sqrt{\frac{3}{2}}\frac{u^3}{(u^4+u^2+1)^{\frac{3}{2}}}$$

$$[\dot{\bar{x}}, \ddot{\bar{x}}, \dddot{\bar{x}}] = \begin{vmatrix} 1 & -\frac{1}{u^2} & -\frac{1}{u^2} - 1 \\ 0 & 2/u^3 & 2/u^3 \\ 0 & -6/u^4 & -6/u^4 \end{vmatrix} = 1\left[-\frac{12}{u^7} + \frac{12}{u^7}\right] = 0$$

$$\text{Torsion } \tau = \frac{(\dot{\bar{x}}, \ddot{\bar{x}}, \dddot{\bar{x}})}{(\dot{\bar{x}} \times \ddot{\bar{x}}).(\dot{\bar{x}} \times \ddot{\bar{x}})} = \frac{0}{\frac{12}{u^6}} = 0$$

$$\therefore \ \tau = 0$$

Problem: 4

Show that necessary & sufficient condition for a plane curve is $\tau = 0$ at all points.

Solution:

Necessary: **Assume that the space curve is a plane curve.**

Now this plane is the osculating plane at all points.

$\therefore$ The binomial $\bar{b}$ is $\perp_r$ to the osculating plane.

$\therefore$ At all the points, $\bar{b}$ is fixed (constant)

$$\frac{d\bar{b}}{ds} = 0$$

$$\text{(i.e.) } -\tau \, \bar{n} = 0$$

$$\text{(i.e.) } \quad \tau = 0 \qquad (\because \bar{n} \neq 0)$$

Conversely:

Assume $\tau = 0$ at all points

(i.e.) $-\tau\,\bar{n} = 0$

(i.e.) $\frac{d\bar{b}}{ds} = 0$

$\therefore \bar{b}$ is a constant vector.

Suppose the space curve is given by the equation $\bar{x} = \bar{x}(s)$. Then

$$(\bar{x}\,.\,\bar{b})' = \bar{x}'\,.\,\bar{b} + \bar{x}\,.\,\bar{b}'$$

$$= 0 + 0 = 0$$

$$\therefore\ (\bar{x}\,.\,\bar{b}) = \text{constant}$$

$$\bar{x}\ \perp_r\ \bar{b}$$

$\therefore$ The position vector of all points on the space curve is $\perp_r$ to the constant vector $\bar{b}$.

This is possible only when the curve lies on the plane and $\bar{b}$ is $\perp_r$ to the plane.

(i.e.) The space curve is a plane curve.

Sec: 1.6

Formulae of Frenet (or) Serret - Frenet Formulae

The formulae of Frenet are

(i) $\frac{d\bar{t}}{ds} = k\bar{n}$

(ii) $\frac{d\bar{n}}{ds} = \tau\bar{b} - k\bar{t}$

(iii) $\frac{d\bar{b}}{ds} = -\tau\bar{n}$

Proof:

[We have proved (i) in Sec:1.4 and (iii) in Sec: 1.5]

(ii) w.k.t $\bar{n}\,.\,\bar{n} = 1$

Diff. w. r. t 's',

$$\bar{n}'\,.\,\bar{n} + \bar{n}'\,.\,\bar{n} = 0$$

$$2\bar{n}'\,.\,\bar{n} = 0$$

$$\bar{n}'\,.\,\bar{n} = 0$$

$$\therefore\ \bar{n}' \perp \bar{n}$$

$\therefore$ $\bar{n}'$ lies in the plane of $\bar{b}$ & $\bar{t}$.

(i.e.) $\bar{n}'$ can be written as a linear combination of $\bar{t}$ and $\bar{b}$.

Let $\bar{n}'' = a_1\,\bar{t} + a_2\bar{b} \longrightarrow (1)$ where a_1 & a_2 are Scalar.

W.k.t. $\bar{t}\,.\,\bar{n} = 0$

Diff. w.r.t. 's'

$$\bar{t}''.\bar{n} + \bar{t}\,.\,\bar{n}'' = 0$$

$$k\bar{n}\,.\,\bar{n} + \bar{t}\,.\,\bar{n}' = 0$$

$$k + \bar{t}\,.\,\bar{n}'' = 0 \qquad \because (\bar{n}\,.\,\bar{n} = 1)$$

$$\therefore\ \bar{t}\,.\,\bar{n}'' = -k \longrightarrow (2)$$

Take dot product with $\bar{t}$ in equation (1)

$$(1) \Rightarrow \bar{t}\,.\,\bar{n}'' = a_1\,\bar{t}.\bar{t} + a_2\,\bar{t}.\bar{b}$$

$$-k = a_1 + a_2(0)$$

$$\text{(i.e.) } a_1 = -k$$

w.k.t $\bar{n}\,.\,\bar{b} = 0$

Diff. w.r.t 's'

$$\bar{n}''\,.\,\bar{b} + \bar{n}\,.\,\bar{b}' = 0$$

$$\bar{n}''\,.\,\bar{b} + \bar{n}\,(-\tau\,\bar{n}) = 0$$

$$\bar{n}''\,.\,\bar{b} - \tau = 0$$

$$\bar{n}''\,.\,\bar{b} = \tau \longrightarrow (3)$$

Take dot product with b in equation (1)

$$(1) \Rightarrow \bar{n}'\,'\,' . \bar{b} = a_1 \bar{t} . \bar{b} + a_2 \bar{b} . \bar{b}$$

$$\tau = a_1(0) + a_2 \qquad \because by\ (3)$$

$$\Rightarrow a_2 = \tau$$

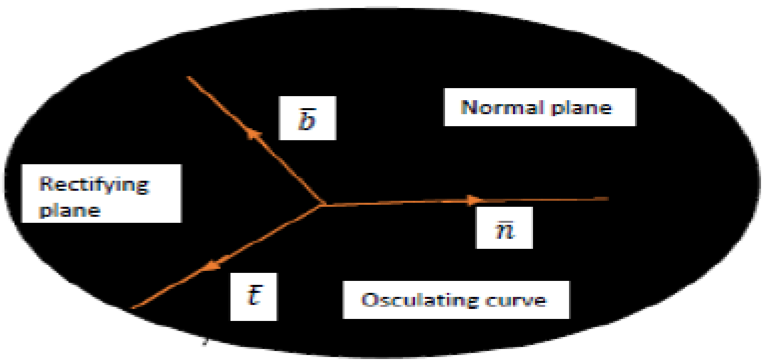
$$(1) \Rightarrow \bar{n}'\,'\,' = -k\,\bar{t} + \tau\,\bar{b}$$

$$\text{(i.e.) } \frac{d\bar{n}}{ds} = \tau\,\bar{b} - k\bar{t}$$

Note: Curvature & torsion are also known as 1st and 2nd curvature, respectively. The space curve is known as curve of double curvature.

Equation of the Osculating Plane:

The plane containing the tangent and principle normal is called an osculating plane. It's equation is given by $(\overline{y} - \overline{x}).\overline{b} = 0$



Normal Plane:

It is a plane containing principal normal and binormal. It's equation is given by $(\overline{y} - \overline{x}).\overline{t} = 0$

Rectifying Plane:

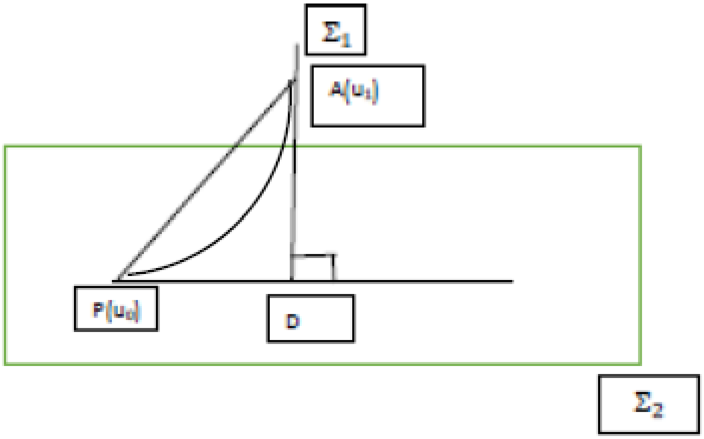
It is a plane containing binormal & tangent. It's equation is given by $(\overline{y} - \overline{x}).\overline{n} = 0$

Sec: 1.7 Contact

Introduction:

Instead of stating that 2 curves having a certain number of points in common, we can also state that they have a contact of certain order.

Contact:



Let two curves Σ_1 *and* Σ_2 have a regular point P in common. Take a point A on Σ_1 near P.

Let AD be its distance to Σ_2. Then Σ_2 has a contact of order n with Σ_1 at P, when for $A \to P$ along Σ_1 .

$$\lim \frac{AD}{(AP)^K} \neq 0 \text{ is finite for } k = n + 1$$

$$\text{But } \lim \frac{AD}{(AP)^k} = 0 \text{ for } k = n$$

$$\text{(i.e.) } AD = 0\,(\,(AP)^K),\ k = 1, 2, \dots \dots \dots n$$

Result:

Necessary and sufficient condition that the surface has a contact of order n at P with a curve are that P, the following relation hold.

$$f(u) = 0 = f'(u) = \cdots = f^n(u),\ f^{(n+1)}(u) \neq 0$$

Proof:

Let Σ_1 be a curve given by $\bar{x}(u)$. Let Σ_2 be a surface given by $F(x, y, z) = 0$ where F_x, F_y, F_z not all zero.

we can make use of the fact that the distance AD of the point $A(x_1, y, z_1)$ near P is of the same order as $F(x_1, y, z_1)$.

The function $f(u)$ can be expressed as $f(u) = F(x(u), y(u), z(u))$.

[it is obtained by substituting x_i of the curve Σ_1 on $F(x, y, z)$]

Let $f(u)$ near the point $P(u = u_o)$ have finite derivatives $f^i(u), i = 1, 2, \dots, n + 1$

Thus if we take $(u = u_1)$ at A and $(h = u_1 - u_o)$ then $\exists$ a Taylor development of $f(u)$ in the form

$$f(u_1) = f(u_o) + \frac{h}{1!} f'^{(u_o)} + \frac{h^2}{2!} f''(u_o) + \cdots + \frac{h^{n+1}}{(n+1)!} f^{n+1}(u_o) + O(h^{n+1}).$$

Here $f(u_o) = 0$. Because $P(u_o)$ is common to Σ_1 and Σ_2, since P lies on Σ_2, $f(u_1)$ is of $O(AD)$.

Hence we get a N and S condition for a curve has a contact of order n with the surface at P as,

$$f(u) = 0 = f'(u) = \cdots = f^n(u),\ \ f^{(n+1)}(u) \neq 0$$

Note:

1) If Σ_1 is the curve $\bar{x}(u)$ and Σ_2 is the another curve given by $F_1(x, y, z) = 0$ & $F_2(x, y, z) = 0$ then N & S condition for a contact of order n at P between the curves are

$$f_1(u) = 0 = f_1^{\prime(u)} = \cdots = f_1{}^{n}(u) = 0$$

$$f_2(u) = 0 = f_2^{\prime(u)} = \cdots = f_2{}^{n}(u) = 0 \ \&$$

at least one of the 2 derivatives $f_1^{(n+1)}(u), f_2^{(n+1)}(u)$ does not vanish at P.
where $f_1(u) = \ F_1(x(u), y(u), z(u))$ & $\ f_2(u) = \ F_2(x(u), y(u), z(u))$

2) The figure Σ_1 & Σ_2 have a contact of order n will have (n+1) consecutive points in common.

3) The tangent has a contact of order one with the curve.

4) The osculating plane & the osculating circle have a contact of order two with the curve.

Osculating sphere:

A sphere passing through Four consecutive points on a ~~sphere~~ Space curve is called the Osculating sphere.

To find the centre of the osculating sphere:

The equation of the sphere is given by $(\overline{X}-\overline{C}).(\overline{X}-\overline{C})-r^2=0$, where $\overline{X}$ is the generic point of the sphere, $\overline{C}$ is the centre and r is the radius.

Let $f(s)=(\overline{x}-\overline{c}).(\overline{x}-\overline{c})-r^2$ Where $\overline{x}$ is the specific point on the sphere.

Now the equation of the sphere becomes $f(s)=0$.

To find the limiting values of $\overline{c}$ & $\overline{r}$ we must have $f'(s)=0, f''(s)=0, f'''(s)=0$

Now $f'(s)=0 \Rightarrow (\overline{x}'-0)(\overline{x}-\overline{c})+(\overline{x}-\overline{c})(\overline{x}'-0)=0 \quad (\because \overline{x}'=\overline{t}\,)$

$$\Rightarrow 2(\overline{x}-\overline{c})\overline{t}=0$$

$$\Rightarrow (\overline{x}-\overline{c})\overline{t}=0 \qquad \text{------------------(1)}$$

$$f''(s) = 0 \Rightarrow (\bar{x}' - 0)\bar{t} + (\bar{x} - \bar{c})\bar{t} = 0$$

$$\Rightarrow \bar{t}.\bar{t} + (\bar{x} - \bar{c}).k.\bar{n} = 0$$

$$\Rightarrow 1 + (\bar{x} - \bar{c}).k.\bar{n} = 0$$

$$\Rightarrow (\bar{x} - \bar{c}).k.\bar{n} = -1$$

$$\Rightarrow (\bar{x} - \bar{c}).\bar{n} = \frac{-1}{k}$$

$$\Rightarrow (\bar{x} - \bar{c}).\bar{n} = -R \quad \text{-------------------(2)} \qquad \text{where } R = \frac{1}{k}$$

$$f'''(s) = 0 \Rightarrow (\bar{x}' - 0)\bar{n} + (\bar{x} - \bar{c})\bar{n}' = -R'$$

$$\Rightarrow \bar{t}.\bar{n} + (\bar{x} - \bar{c})(\tau\bar{b} - k\bar{t}) = -R'$$

$$\Rightarrow 0 + (\bar{x} - \bar{c})\,\tau\bar{b} - (\bar{x} - \bar{c})\,k.\bar{t} = -R'$$

$$\Rightarrow (\bar{x} - \bar{c})\,\tau\bar{b} - 0 = -R' \qquad (\because \text{from } (1))$$

$$\Rightarrow (\bar{x} - \bar{c})\,\bar{b} = -\frac{R'}{\tau}$$

$$\Rightarrow (\bar{x} - \bar{c})\,\bar{b} = -R'T \quad \text{------------(3)} \qquad \because \frac{1}{\tau} = T$$

(1) gives $(\overline{x}-\overline{c}) \perp_r \overline{t}$. $\therefore \overline{x}-\overline{c}$ lies in $\overline{n}$ & $\overline{b}$

Let $(\overline{x}-\overline{c}) = \alpha_1.\overline{x} + \alpha_2.\overline{b}$ -------------------(4)

Take dot product with $\overline{b}$ in (4), we ge

$$\overline{b}.(\overline{x}-\overline{c}) = \alpha_1.\overline{x}.\overline{b} + \alpha_2.\overline{b}.\overline{b}$$

$$-R'T = 0 + \alpha_2.1 \qquad (\because \text{by}(3)$$

$$\alpha_2 = -R'T$$

Take dot product with $\overline{n}$ in (4), we get

$$\overline{n}(\overline{x}-\overline{c}) = \alpha_1\,\overline{n}.\overline{n} + \alpha_2\,\overline{b}.\overline{n}$$

$$-R = \alpha_1.1 + \alpha_2.0 \qquad \because by\ (2)$$

$$\Rightarrow \alpha_1 = -R$$

$$(4) \Rightarrow (\overline{x}-\overline{c}) = -R\overline{n} - R'T\overline{b}$$

$$\therefore \overline{c} = \overline{x} + R\overline{n} + R'T\overline{b}$$

This is the centre of the osculating sphere.

When the curve is of constant curvature then $\frac{1}{k}$ is a constant.

(i.e.) R = constant $\quad \therefore R' = 0$

Now centre of the osculating sphere becomes $\bar{c} = \bar{x} + R\,\bar{n}$ which is the centre of the osculating circle.

Thus, we have "when the curve is of constant curvature, the centre of the osculating sphere coincides with the centre of the osculating circle".

Note: 1 The osculating sphere has a contact of order <u>three</u> with the curve.

Note: 2 The radius of the osculating sphere is $r = \sqrt{R^2 + (R'r)^2}$

Sec: 1.8 Natural Equation

An equation of the curve in the form $\bar{x} = \bar{x}(u)$ depends on a coordinate system. An equation to the curve which does not depend on such a coordinate system is referred to as **natural equation (or) intrinsic equation**

A natural equation to the curve can be given in the form $\bar{k} = \bar{k}(s)$, where $\bar{k}$ is the curvature & s is the arc length. It is possible to change from natural equation to an ordinary equation and vice versa.

Fundamental Theorem for space curve

If two single valued continuous functions $k(s)$ & $\tau(s)$, $s > 0$ are given then $\exists$ only one space curve determined, for its position in space for which s is the arc length, k is the curvature and τ is the torsion.

Proof:

1. **Analytic Proof:**

Assume that the given functions $k(s)$ & $\tau(s)$ are analytic. Then the neighbourhood of the point $s = s_0$, we can use Taylor expansion for $\bar{x}(s)$.

Take $h = s - s_0$, then for the interval $s_1 \leq s \leq s_2$,

$$\bar{x}(s) = \bar{x}(s_0) + \frac{h}{1!}\bar{x}'(s_0) + \frac{h^2}{2!}\bar{x}''(s_0) + \cdots$$

Using the formula of Frenet,

$$\bar{x}' = \bar{t},\ \bar{x}'' = \bar{t}' = k\bar{n},\ \bar{x}''' = k'\bar{n} + k(\tau\bar{b} - k\bar{t}) = -k^2\bar{t} + k'\bar{n} + k\tau\bar{b}$$

Using these values in $\bar{x}(s)$, we have

$$\bar{x}(s) = \bar{x}(s_0) + \frac{h}{1!}\bar{t} + \frac{h^2}{2!}k\bar{n} + \frac{h^3}{3!}(-k^2\bar{t} + k'\bar{n} + k\tau\bar{b}) + \cdots$$

Here all the terms on the R.H.S exists, since the functions $k(s)$ & $\tau(s)$ are analytic.

Thus $\bar{x}(s)$ is uniquely determined in terms of $\bar{x}(s_0), \bar{t}, \bar{n}, \bar{b}$ at $s = s_0$

Thus, the curve is uniquely determined except for a position in space.

2. Non- Analytic Proof:

Suppose the analyticity of $k(s)$ & $\tau\,(s)$ are not assumed. Consider the system of 3 simultaneous diff. equations of first order in α, β, γ as follows

$$\left\{\frac{d\alpha}{ds} = k\beta\,,\ \frac{d\beta}{ds} = \tau\gamma - k\alpha,\ \frac{d\gamma}{ds} = -\tau\beta\right\} \longrightarrow (1)$$

From the theory of differential equations, when k & τ are single valued functions in the given interval, the above differential equations have a unique set of solutions, which assume given values for the arguments.

Using this results, we can find a unique set of solutions $\alpha_1(s)$, $\beta_1(s), \gamma_1(s)$, this solutions assumes the specific values (1, 0, 0) for $s = s_0$.

(i.e.) $\alpha_1(s_0) = 1,\quad \beta_1(s_0) = 0,\quad \gamma_1(s_0) = 0$

///ly $\alpha_2(s_0) = 0,\quad \beta_2(s_0) = 1,\quad \gamma_2(s_0) = 0$

$\alpha_3(s_0) = 0,\quad \beta_3(s_0) = 0,\quad \gamma_3(s_0) = 1$

Now, the system of differential equation given by equation (1) lead to the following result

$$\frac{1}{2}\frac{d}{ds}(\alpha_1^2 + \beta_1^2 + \gamma_1^2) = \frac{1}{2}\{2\alpha_1\left(\frac{d\alpha_1}{ds}\right) + 2\beta_1\left(\frac{d\beta_1}{ds}\right) + 2\gamma_1\left(\frac{d\gamma_1}{ds}\right)\}$$

$$= \frac{1}{2}\,.2\{\,\alpha_1 k\beta_1 + \beta_1(\tau\gamma_1 - k\alpha_1) + \gamma_1(-\tau\beta_1)\,\}$$

$$= \alpha_1\, k\beta_1 + \beta_1\tau\gamma_1 - \beta_1 k\alpha_1 - \gamma_1\tau\beta_1 \quad = 0$$

$$\therefore (\alpha_1^2 + \beta_1^2 + \gamma_1^2) = c, constant.$$

At $s = s_0$, $\alpha_1 = 1, \beta_1 = 0, \gamma_1 = 0$, $1 + 0 + 0 = c \Rightarrow c = 1$

Hence $\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1$

$///^{ly}$ $\alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1$

$\alpha_3^2 + \beta_3^2 + \gamma_3^2 = 1$

Also $\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 1$

$\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 = 1$

$\alpha_3\alpha_1 + \beta_3\beta_1 + \gamma_3\gamma_1 = 1$

Thus we have found 3 mutually $\perp_r$ vectors, $\bar{t}(\alpha_1, \alpha_2, \alpha_3), \bar{n}\ (\beta_1, \beta_2, \beta_3), \bar{b}\ (\gamma_1, \gamma_2, \gamma_3)$ where $\alpha_i, \beta_i, \gamma_i$ are functions of s.

Thus we have infinite number of trihedrons.

Now the equation $\bar{x}(s) = \int_{s_0}^{s} \bar{t}\ ds$ determines a curve for which $\bar{t}, \bar{n}, \bar{b}$ as moving trihedrons, k & τ being its curvature & Torsion and s its arc length.

Hence ∃ a curve c with a given curvature & torsion. We prove this curve is unique. For this, suppose that there is another curve $\bar{c}$ with same $k(s)$ & $\tau\ (s)$.

We will bring the curve $\bar{c}$ to coincide with c. For this let us move the point $s = 0\ of\ \bar{c}$ to the point $s = 0\ of\ c$, such that at this point $\bar{t}, \bar{n}, \bar{b}$ coincide with $\bar{t}, \bar{n}, \bar{b}\ of\ \bar{c}$ & choose this as coordinate axis.

Let $(x_i, \alpha_i, \beta_i, \gamma_i)$ & $(\bar{x}_i, \bar{\alpha}_i, \beta_i, \bar{\gamma}_i)$ denote the corresponding elements of moving trihedron of c & $\bar{c}$ respectively.

This system of (1) hold for $\alpha_i, \beta_i, \gamma_i$ and $\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i$. Now

$$\frac{d}{ds}(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma}) = \alpha\left(\frac{d\bar{\alpha}}{ds}\right) + \bar{\alpha}\left(\frac{d\alpha}{ds}\right) + \beta\left(\frac{d\bar{\beta}}{ds}\right) + \bar{\beta}\left(\frac{d\beta}{ds}\right) + \gamma\left(\frac{d\bar{\gamma}}{ds}\right) + \bar{\gamma}\left(\frac{d\gamma}{ds}\right)$$

$$= \alpha k\bar{\beta} + \bar{\alpha}k\beta + \beta(\tau\bar{\gamma} - k\bar{\alpha}) + \bar{\beta}(\tau\gamma - k\alpha) + \gamma(-\tau\bar{\beta}) + \bar{\gamma}(\tau\beta)$$

$$= 0$$

$$\therefore \ \alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} = constant$$

At $s = s_0,\ \alpha = \bar{\alpha} = 1,\ \beta = \bar{\beta} = 0,\ \gamma = \bar{\gamma} = 0$

$\therefore$ The above equation gives

$$\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} = 1$$
$$\alpha_i\bar{\alpha}_i + \beta_i\bar{\beta}_i + \gamma_i\bar{\gamma}_i = 1,\ i = 1, 2, 3, \ldots$$

Also we have

$$\alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1 \ \& \ \bar{\alpha}_i^2 + \bar{\beta}_i^2 + \bar{\gamma}_i^2 = 1$$

$\Rightarrow$ The vector $(\alpha_i, \beta_i, \gamma_i)$ & $(\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i)$ makes zero angle with one another.

(i.e.) $\bar{\alpha}_i = \alpha_i,\ \bar{\beta}_i = \beta_i,\ \bar{\gamma}_i = \gamma_i$

Hence $\forall\, s,\ \frac{d}{ds}(\bar{x}_i - x_i) = 0$

$$\bar{x}_i - x_i = constant$$

At $s = s_0, \bar{x}_i - x_i = 0 \Rightarrow \bar{x}_i = x_i$

Thus the curve $\bar{c}$ can be made to coincide with curve c. Hence there is only one space curve for the given $k(s)$ & $\tau\,(s)$.

Sec:1.9 Helices

The circular helices is a special case of large class of curves called the cylindrical helices (or) helices (or) curves of constant slope. Helix is defined by the property that the

Let us assume that the curve be of constant slope. (i.e.) Let $\bar{x} = \bar{x}(s)$ be a helix. Let $\bar{a}$ be the unit vector along the fixed line l. Then by definition of helix,

Theorem:1.9.1

Necessary and Sufficient condition that a curve of constant slope is that the ratio of curvature to torsion be constant.

Proof:

Let us assume that the curve be of constant slope. (i.e.) Let $\bar{x} = \bar{x}(s)$ be a helix. Let $\bar{a}$ be the unit vector along the fixed line l. Then by definition of helix,

$$\bar{t}\,.\,\bar{a} = \cos\alpha = constant$$

Diff. w.r.t 's',

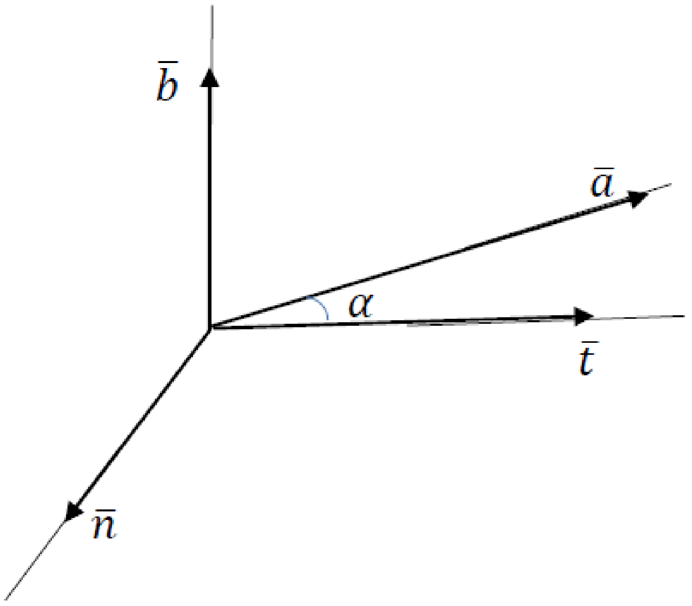
$$\frac{d\bar{t}}{ds}\,.\,\bar{a} = 0$$

By Frenet formulae,

$$k.\bar{n}.\bar{a} = 0$$

(i.e.) $\bar{n}\,.\,\bar{a} = 0$

(i.e.) $\bar{a}$ *is* $\perp_r$ *to* $\bar{n}$.



Hence $\bar{a}$ may be parallel to the rectifying plane.

∴ we can write

$$\bar{a} = \bar{t}\cos\alpha + \bar{b}\sin\alpha$$

Diff. w.r.t 's',

$$\frac{d\bar{a}}{ds} = \bar{t}'\cos\alpha + \bar{b}'\sin\alpha$$

$$0 = k.\bar{n}\cos\alpha - \tau.\bar{n}\sin\alpha$$

(i.e.) $k\cos\alpha = \tau\sin\alpha$

$$\frac{k}{\tau} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = Constant\,.$$

Thus for a curve of constant slope $\frac{k}{\tau}$ is constant.

Conversely assume that $\frac{k}{\tau} = constant$ at all points for a given curve $\overline{x} = \overline{x}(s)$.

$$\frac{k}{\tau} = \tan\alpha \;\; (\alpha \; is \; constant)$$

$$\frac{k}{\tau} = \frac{\sin\alpha}{\cos\alpha}$$

$$k\cos\alpha = \tau\sin\alpha$$

Multiply by $\overline{n}$ on both sides, we get

$$k\overline{n}\cos\alpha = \tau\,\overline{n}\sin\alpha$$

$$k\overline{n}\cos\alpha - \tau\,\overline{n}\sin\alpha = 0$$

$$\Rightarrow t^{-1}\cos\alpha + b^{-1}\sin\alpha = 0$$

$$\frac{d}{ds}(\overline{t}\cos\alpha + \overline{b}\sin\alpha) = 0$$

$$\overline{t}\cos\alpha + \overline{b}\sin\alpha = constant$$

(i.e.) $\overline{t}\cos\alpha + \overline{b}\sin\alpha = \overline{a}$ [$\therefore$ $\overline{a}$ *is a constant vector*)

Take dot product with $\overline{t}$ we get

$$\overline{t}\,\overline{t}\cos\alpha + \overline{t}\;\overline{b}\sin\alpha = \overline{a}\;\overline{t} \quad \Rightarrow \cos\alpha = \overline{a}\;\overline{t}$$

$$\overline{t}\;\overline{a} = \cos\alpha = constant.$$

By definition of helix, the given curve $\overline{x} = \overline{x}(s)$ is a helix.

(i.e) The curve is of constant slope.

Theorem: 1.9.2

The Projection of a helix on a plane perpendicular to its axis has its principal normal parallel to corresponding principal normal of the helix and its corresponding curvature is $k_1 = \text{k}\ cosec^2\alpha$

Proof:-

If we project the helix $\overline{x}\ (s)$ on a plane perpendicular to $\overline{a}$, then the projection $\overline{x_1}(s)$ has the equation

$$\overline{x_1} = \overline{x} - (\overline{x} - \overline{a})\ \overline{a}$$

Now diff w.r.t'S', we have

$$\frac{d\overline{x_1}}{ds} = \frac{d\overline{x}}{ds} - (\frac{d\overline{x}}{ds}\ .\overline{a})\ \overline{a}$$

$$= \overline{t} - (\overline{t}\ .\overline{a})\ \overline{a}$$

$$= \overline{t} - (\cos\alpha)\overline{a} \quad \rightarrow [1]$$

Now they arc length in $\overline{x_1}$ is given by

$$ds_1{}^2 = d\overline{x_1}\ .\ d\overline{x_1}$$

$$= (\overline{t} - \cos\alpha\ \overline{a})\ (\overline{t} - \cos\alpha\ \overline{a})\ ds^2 \quad (\text{ since by [1]}$$

$$= \{\ \overline{t}\ \overline{t}\ - \overline{t}\cos\alpha\ \overline{a}\ - \overline{t}\cos\alpha\ \overline{a} + cos^2\alpha\ \overline{a}\ \overline{a}\ \}\ ds^2$$

$$= \{1 - 2\ (\overline{t}\ .\ \overline{a})\ \cos\alpha + cos^2\alpha\ \}\ ds^2$$

$$= \{1 - 2\cos^2\alpha + \cos^2\alpha\}\, ds^2$$

$$ds_1^{\,2} = \{1 - \cos^2\alpha\}\, ds^2$$

$$ds_1^{\,2} = \sin^2\alpha \; ds^2$$

(i.e) $ds_1 = \sin\alpha \, ds \quad \dashrightarrow [2]$

Now $\dfrac{d\overline{x_1}}{ds_1} = \dfrac{d\overline{x_1}}{ds} \cdot \dfrac{ds}{ds_1}$

$$= (\bar{t} - \cos\alpha\,\bar{a}) \cdot \frac{1}{\sin\alpha}$$

$$= (\bar{t} - \bar{a}\cos\alpha) \cdot \operatorname{cosec}\alpha$$

Again diff with respect to s_1, we have

$$\frac{d^2\overline{x_1}}{ds_1^{\,2}} = \frac{d\bar{t}}{ds_1} \operatorname{cosec}\alpha$$

$$= \frac{d\bar{t}}{\sin\alpha\, ds} \operatorname{cosec}\alpha \quad \text{(Since by [2]}$$

$$= \frac{d\bar{t}}{ds} \operatorname{cosec}^2\alpha \quad = \bar{t}' \operatorname{cosec}^2\alpha$$

$$k_1 \bar{x} = k\, \bar{x} \operatorname{cosec}^2\alpha$$

$$k_1 = k \operatorname{cosec}^2\alpha$$

Example -1 Circular Helix

If a helix has a constant curvature, then its projection on the plane perpendicular to its axis is a plane curve of constant curvature, hence a circle. The helix lies on a cylinder of revolution and is therefore a circular helix.

Example-2 Spherical helix

A spherical helix projects on a plane perpendicular to its axis in an arc of an epicycloid. If a helix lies on a sphere of radius r, then we have the equation $R^2 + (R'T)^2 = r^2$ where r is a constant with $\frac{k}{\tau}$ = constant = tanα.

The above equation together with k = τ tan α, gives after eliminating of τ.

$$r^2 = R^2 \left(1 + R'^2 \tan^2\alpha\right)$$

$$r^2 - R^2 = R^2 R'^2 \tan^2\alpha$$

$$\frac{r^2 - R^2}{R^2} = R'^2 \tan^2\alpha \ .$$

$$\text{(i.e) } \pm\sqrt{\frac{r^2-R^2}{R^2}} = \sqrt{R'^2 \tan^2\alpha}$$

$$\pm\sqrt{\frac{r^2-R^2}{R^2}} = \frac{dR}{ds}\tan\alpha$$

$$\pm\frac{ds}{\tan\alpha} = \frac{RdR}{\sqrt{r^2-R^2}}$$

$$\frac{RdR}{\sqrt{r^2-R^2}} = \pm\cot\alpha\, ds$$

$$-\frac{1}{2}\,\frac{d(r^2-R^2)}{\sqrt{r^2-R^2}} = \pm\cot\alpha\, ds$$

Integrating we get

$$-\sqrt{r^2-R^2} = \pm\, s\cot\alpha$$

Squaring,

$$r^2 - R^2 = s^2\cot^2\alpha$$

$$r^2 = R^2 + s^2\cot^2\alpha$$

$$r^2 = R^2\frac{\sin^2\alpha}{\sin^2\alpha} + s^2\frac{\cos^2\alpha}{\sin^2\alpha}$$

$$r^2 = \frac{R^2\sin^2\alpha + s^2\cos^2\alpha}{\sin^2\alpha}$$

$$r^2\sin^2\alpha = R^2\sin^2\alpha + s^2\cos^2\alpha$$

Multiply by $\sin^2\alpha$ *we get*

$$r^2\sin^4\alpha = R^2\sin^4\alpha + s^2\sin^2\alpha\cos^2\alpha \quad \to [1]$$

We know that

$$k_1 = k\,cosec^2\,\alpha$$

$$R_1 = R\sin^2\alpha$$

$$R_1{}^2 = R^2\ \sin^4\alpha \qquad \dashrightarrow [2]$$

and

$$S_1 = S\sin\alpha$$

$$S_1{}^2 = S^2\ \sin^2\alpha \qquad \dashrightarrow [3]$$

Using [2] and [3] in equation [1], we get

$$r^2\sin^4\alpha = R_1{}^2 + S_1{}^2\ \cos^2\alpha$$

Since $\cos^2\alpha < 1$ *, the above equation represents an epicycloid.*

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